

Computational Music Science



Octavio Alberto Agustín-Aquino
Julien Junod
Guerino Mazzola

Computational Counterpoint Worlds

Mathematical Theory, Software,
and Experiments

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Computational Music Science

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Mathematical Theory, Software,
and Experiments

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Preface

The mathematical theory of counterpoint developed by Guerino Mazzola around 1984 was originally aimed at simulating the composition rules described in Johann Joseph Fux's *Gradus ad Parnassum* [29]. It soon came out that the algebraic apparatus used in this model could also serve to define entirely new systems of rules for composition. Those are generated by new choices of consonances and dissonances, which in turn lead to new restrictions governing the succession of intervals.

Discovering new systems of counterpoint in the usual twelve-note chromatic gamut was a first demonstration of the model's potential for variation. In this book we propose to pursue this exploration of exotic counterpoint worlds, along two axes. First, a *longitudinal* one where we vary the division of the octave and show how macro- and microtonal consonances are embedded into a sequence, up to the pitch continuum. In a second, more *cross-sectional* perspective, we will see how counterpoint structures can be mapped across different systems of the same or different temperaments. This work originated in two PhD dissertations, which were merged to summarize these recent developments in the field of mathematical counterpoint theory: *Extensiones microtonales de contrapunto* was submitted to the Universidad Nacional Autónoma de México in 2011 by Octavio Alberto Agustín Aquino, and *Counterpoint Worlds and Morphisms* to the Universität Zürich in 2010 by Julien Junod. This book is also the first comprehensive exposition of this counterpoint model, bringing together many aspects of the subject previously published in different places, in particular *Geometrie der Töne* [55] and the *Topos of Music* [57]. Guerino Mazzola, the advisor for both theses, joined Octavio and Julien in this project with his recent reformulation of the theory in terms of gestures.

The counterpoint described by the model is the first-species case: two voices playing simultaneously. While handling only the most basic case, we believe that it already contains all the fundamental principles of counterpoint and that higher species only provide a variation of the basic rules. Simplicity has the pedagogical advantage of showing the essential structure of the model without hiding it behind complex configurations.

The explicit nature of counterpoint rules has always attracted computer scientists, and it is very tempting to use them as constraints in algorithmic composition. The

first example of a computer-generated composition, Lejaren Hiller's *Illiac Suite*, dates back to the 1950s.¹ It is not our aim to simulate faithfully vocal polyphonies of the Renaissance. The counterpoint model raises a far-deeper-reaching question: counterpoint has a centuries-long history and, despite its limitations, has proved to be very successful in Western music history. Now, our model tells us that this system is only one instance among many others of a more general structure. So what is the musical value of the other systems? Is the traditional Fuxian system really the best one? Do other worlds as effective for composition exist? And how would such an exotic counterpoint sound? It is our desire that this theory opens up new territories for musical composition, not by starting everything from scratch, but by exporting proven principles.

Chapter 1 provides some justifications for our choice to diverge from standard counterpoint theory. Chapter 2 briefly describes how the model is constructed for any equally tempered scale with an even number of tones, and the third chapter specializes the results to the twelve-tone scale, and a series of further peculiarities that arise in this setup. Chapter 4 introduces the notion of counterpoint world, plus the required graph theory and notation setting the stage for the fifth chapter, which shows under which conditions and how the contrapuntal structure can be morphed from one world to another.

The sixth chapter is essentially independent from the rest, and constitutes a user's guide to the Bolly rubettes, which allow musicians to perform experiments composing traditional and exotic counterpoint and morphing it between worlds.

The following two chapters study with more detail the combinatorial aspects of the model and how consonances can be progressively embedded between tunings.

The last pair of chapters introduce the reader to categorical gesture theory and its consequences for counterpoint, an exciting and novel development that, together with the possibility of infinite sets of consonances and dissonances, announce the notion of continuous counterpoint.

A good command of algebra is necessary to understand the construction of the model, which uses cyclic groups and tangent spaces to describe interval classes. Graph theory plays a role in describing how counterpoint worlds can be morphed into one another and in the proofs of some theorems. While the reader may need some mathematical training to follow the theoretical developments, a layman should be able to apply it for composing music, i.e., using software tools to compose and play with the rules in different systems.

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Montparnasse and
Minneapolis

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Guerino Mazzola
December 2014

¹ Premiered on August 9th, 1956, the score of these *experiments* was published in 1957 [39]. The algorithmic process was first described in [40] and more thoroughly explained in [41], which also contains a reproduction of the original score in its appendix on page 181.

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Chapter 1

Prolegomena on Counterpoint

In Europe, during the Middle Ages, a set of principles intended to organize the concert of many voices emerged and matured; the final result of this process is termed *counterpoint*, and it is the main concern of this book. In this chapter we shall briefly discuss its meaning and some highlights of its historical roots, in order to justify the necessity, utility, and assumptions of the mathematical model that has been developed by the authors. For a detailed discussion of the history of counterpoint, see [10, 43, 78].

1.1 Counterpoint's Many Voices

As we have already stated, one of the key ingredients of counterpoint is its multiplicity of voices or *polyphony*. Frequently, this can be taken literally, since most of the music that shaped counterpoint was of vocal nature. Thus, many of its principles take into account what is feasible for an ensemble of singers to perform.

The vocal origin of counterpoint is kind of a mixed blessing. On the one hand, it allowed both composers and performers to evolve certain optimal structures (with respect to appropriate parameters and restrictions) through improvisation and artificial selection. But on the other hand, it seriously limited their imaginations, so they didn't experiment with a wider palette of possibilities, thus laying the foundations of creativity walls (as discussed in [62, chapter 11]) that hindered further evolution of this important compositional tool.

In order to overcome these limitations, a precise mathematical model is required. This may seem problematic, but it is quite the opposite: The use of mathematical thinking allows us to be more aware of the assumptions and scope of traditional counterpoint and, by manipulating them, to suggest new developments. Our points of departure, of course, have been questioned several times. Here is a list of the main objections:

1. How do you justify the selection of consonances? Why do you consider the perfect fourth as a dissonance?

2. Why do you understand counterpoint not as the confrontation of two melodic lines, but as a succession of intervals?
3. Why is first-species counterpoint the first target of your theory?

In the following sections we give our answers to these questions, and in the last one we identify some walls and show how to debilitate them.

1.2 Consonances, Dissonances, and the Fourth

In the early stages of counterpoint, any interval was allowed as separation between the voices, as long as the first and last ones were unisons [93, p. 65]. Gradually, minor and major seconds were abandoned as consonances; tritones and sevenths quickly followed.

The perfect fourth, represented by a simple frequency ratio $\frac{4}{3}$, is acoustically a consonance and as such was used in the early polyphony of the *organum*: two voice improvisation at a distance of perfect fourths [23, p. 33]. This is exemplified in the remarkable treatise *Musica enchiridias*.¹ Nevertheless, it gradually lost this quality, as stated in the anonymous *Discantus positio vulgaris* or Garlandia's *De mensurabilis musica*, from the thirteenth century, in which the fourth is less consonant than the unison, fifth and octave. One hundred years later, the process is completed, and most writings treat the fourth essentially as dissonant, either in practice or in theory, as is the case of the *Ars contrapunctus secundum Phillipum de Vitriaco* and *Ars contrapunctus secundum Johannem de Muris* [94]. It remained a dissonant and inessential interval even during the Baroque period [46, p. 32], because of certain incompleteness associated with the interval, which resolves to a third.²

While it is true that there are exceptions to the consensus of classifying the perfect fourth as a dissonance (like the notable cases of Tinctoris and Zarlino), it is also true that the practical and theoretical treatment of this interval is, at best, ambiguous; this is a feature undesirable in a predictive model of any kind. The fact that the model we will introduce recovers the most important features of counterpoint without resorting to declaring the fourth a consonance favors the conclusion that it is indeed an unnecessary hypothesis.

1.3 Point Against Point

Another fundamental feature of counterpoint stems from its own name: the placement of a note against a note. This “opposition” is considered a source of tension,

¹ This work, furthermore, considers the pitches a fifth apart to have the same quality [16, p. 155]. This is rather interesting since, from this viewpoint, the set of consonances in Renaissance counterpoint is what is called in mathematics a *monoid* (see Proposition 3.1).

² Later (Example 2.2) we will discover that, in a very precise mathematical way, the “opposite” of a perfect fourth is a minor third.

although such a feature in the vertical domain is difficult to justify, for the distance between the voices should preferably be a consonant interval.

The classical explanation of this “paradox” is that the use of contrary motion of the voices (if one ascends, the other descends, and vice versa) is the real source of differentiation and confrontation between the voices. But the irreflexive application of this principle results in a behavior of voices as predictable or dependent as for parallel motion. Furthermore, many parallel motions are acceptable in classical counterpoint, including any number of consecutive thirds (!).

As is noted by some historians and theorists, the “contra” in counterpoint also comes from the *horizontal* juxtaposition of intervals [78, p. 34], [93, p. 66], fourth against fifth (when the fourth was regarded as consonant) or perfect against imperfect. This is a key observation for the success of the theory that we will describe in subsequent chapters, for we will define an appropriate deformation of consonant intervals into dissonances that affords the desired contrast between them. Furthermore, it is an important result in gesture theory (the so-called *Escher theorem*, see Proposition 9.2) that we can concentrate in the horizontal or the vertical dimension of counterpoint in any order we wish, for both viewpoints are essentially equivalent.

1.4 First-Species Counterpoint

The simplest type of contrapuntal composition is what Johannes Tinctoris calls *contrapunctus simplex*, which consists of two voices performing notes of equal duration simultaneously [43, p. 11]. This kind of counterpoint, which is called *first-species* counterpoint by Johann Fux [30], certainly has many pedagogical advantages, and it is the most elementary setup that allows a non-trivial analysis. But it also had an important practical value when counterpoint was a major architectural device in music: It was expected for trained musicians, given the melody of a voice (the so-called *cantus firmus*), to be able to improvise a first-species second voice, or *discantus*, in what was called *contraponto alla mente* or *super librum cantare* [12, p. 169], [43], [48].

Nonetheless, even if a first-species composition constitutes neither an interesting challenge nor a sublime piece of art, it lies in the core of more refined types of contrapuntal work. Sachs, for example [78, p. 123], considers it as the kernel of the theory during the fourteenth and fifteenth centuries, while the rest are just gradual elaborations. At least for two-part writing, this is confirmed by Fux [30], for his *Gradus ad Parnassum* does not explicitly introduce further rules for other species, save for metrical or melodic considerations or forbidding procedures that surreptitiously mask violations of first-species counterpoint. Therefore, an accurate model of the elementary first-species is a natural and obligatory first step toward a more general and useful mathematical counterpoint theory.

1.5 Three Creativity Walls

The fact that an acoustically consonant interval was cast out as a dissonance, leaving a stark separation of consonant and dissonant intervals in parts of equal size, is, to say the least, difficult to explain by pure psychoacoustic rationales.

Furthermore, this bipartition is just one among 924 possibilities,³ naively ignoring transpositions, inversions, or any other musically meaningful transformation that renders some of them equivalent. Why was it preferred? Once we justify the musical transformations that make sense for counterpoint, we will discover that the other five bipartitions are just as useful as the classical consonance/dissonance dichotomy for mathematically equivalent theories. Moreover, it is possible to define musically meaningful transformations of the contrapuntal rules between these counterpoint *worlds*; this morphing of contrapuntal compositions has been implemented as an extension of the RUBATO, as we will show in Chapters 4 and 5.

This is also a good moment to explain why we will restrict our discussion to equal temperaments. Uniform spaces between the tones of the scale is certainly a very convenient and simplifying hypothesis, but it is not *a priori* essential. It turns out that, at least for the case of Vogel's just intonation, one of the symmetries of a translation of the C-chromatic scale is precisely the same one that relates consonances and dissonances in the tempered scale in our model, once the appropriate identifications are done (see [57, chapter 24] for further details). Although the consequences of this fact constitute an open problem, they at least suggest that the main features of counterpoint are not essentially linked to any particular tuning. It is also important to note that equal temperament facilitates the use of transformations that enrich the contrapuntal practice; it is not by coincidence that the crystallization of counterpoint principles and the popularization of the equi-tempered scale were contemporaneous.

Finally, the whole range of equal temperaments is a realm where little or no theory or practice has been done. Except for the isolated cases of Alois Hába, Julián Carrillo, and a few others, scant progress has been made in the direction of macro- and microtonality for counterpoint. Once the feasibility of the model is proved for these setups, the model carries over in a natural and direct way, with very surprising and even counterintuitive results. One of them is that it is possible to halve tones preserving the consonances of the original scale, and even distributing the consonances in such a way that there is a consonance (and a dissonance too) arbitrarily close to *any* interval.

³ We have $\binom{12}{6} = \frac{12!}{6!6!} = 924$.

Chapter 2

First-Species Model

Here we present the full first-species model for a $2k$ -tone equal temperament. In particular, we explain what a *counterpoint dichotomy* and a *contrapuntal symmetry* is, how the rules of counterpoint are deduced from these concepts, and how to do the relevant calculations.

2.1 Dichotomies

A significant part of [57] proves that modules are appropriate mathematical objects for the aim of capturing essential features of musical objects; we will also take them as the natural ambient of our theory. The reader can consult the mathematical appendix, Section A.4, for a summary about elementary module theory. In what follows, we will always suppose that the module M is defined over a commutative ring R .

Definition 2.1. *Let M be R -module. A marked dichotomy X is a subset of M such that $|X| = |X^c|$.*

We denote with $MiD(M)$ the set of all marked dichotomies of M . Note that when M is finite, its cardinality must be an even number for our definition to make sense. We shall also use this definition in case M is any set since the module structure does not really matter.

Example 2.1. In the \mathbb{Z} -module \mathbb{Z} , the set of even integers is a marked dichotomy. If \mathbb{Z}_2 is the field of two elements, then $X = \{(0,0), (0,1)\}$ is a dichotomy in the \mathbb{Z}_2 -module \mathbb{Z}_2^2 .

Remember that the \mathbb{Z} -module \mathbb{Z}_{2k} parametrizes pitch classes in the equally tempered $2k$ -chromatic scale. We are interested in the intervals between pitch classes, i.e., the differences $x - y$ of pitch classes x and y in \mathbb{Z}_{2k} . Mathematically speaking, the result $x - y$ is again an element of \mathbb{Z}_{2k} , but in musical terms it is understood as

an interval. To make this subtlety more evident, we consider the module $\varepsilon.\mathbb{Z}_{2k}$ of intervals in \mathbb{Z}_{2k} , whose elements are intervals in the pitch class module \mathbb{Z}_{2k} . The choice of the notation will become clear soon.

Definition 2.2. A marked dichotomy of intervals is a marked dichotomy of the \mathbb{Z} -module $\varepsilon.\mathbb{Z}_{2k}$.

A marked interval dichotomy is sometimes denoted by (X/X^c) , in order to make explicit the complement.

As we mentioned in the first chapter, we will regard some transformations of a given dichotomy X as essentially equivalent to the original. This idea is captured by the notion of a group acting on a set: in short, this means we have a set X endowed with a group of symmetries G that transform X . The *orbit* of an element $k \in X$ with respect to the action of G is the set of all possible outcomes of a symmetry applied to k (more formal definitions can be found in Section A.3 of the mathematical appendix).

The first of these transformations for marked interval dichotomies that we will consider is the simple act of switching from the one part of a dichotomy to its complement, which defines the following action of \mathbb{Z}_2 over $MiD(M)$:

$$\begin{aligned} ?^c : \mathbb{Z}_2 \times MiD(M) &\longrightarrow MiD(M), \\ (i, X) &\longmapsto X^{ic}; \end{aligned}$$

here X^{ic} stands for the operation of taking i times the complement of X .

The process of reversing the roles of consonance and dissonance is not foreign to musical practice. On a small scale, we have seen in the first chapter this change specifically for the interval of a fourth and in the recent idea of dissonant counterpoint [20, p. 35].

Another kind of transformation comes from the action of the general affine group $\overrightarrow{GL}(\mathbb{Z}_{2k})$, which is the group that consists of these bijective functions

$$\begin{aligned} T^u \cdot v : \mathbb{Z}_{2k} &\longrightarrow \mathbb{Z}_{2k}, \\ x &\longmapsto T^u \cdot v(x) = u + v(x), \end{aligned}$$

where $u \in \mathbb{Z}_{2k}$ defines a shifting or transposition, and $v \in GL(\mathbb{Z}_{2k})$ is a linear isomorphism on \mathbb{Z}_{2k} . Please note that we are doing a slight abuse of notation, writing $v(x)$ for the multiplication $v \cdot x$ by the invertible element $v \in \mathbb{Z}_{2k}^\times$. It is worthwhile to state that for the twelve-tone scale there are four invertible elements—1, 5, 7 and 11—as is well known from pitch class theory.

Another way of understanding $\overrightarrow{GL}(\mathbb{Z}_{2k})$ is as the set $\mathbb{Z}_{2k} \times \mathbb{Z}_{2k}^\times$ with the group operation

$$(u_1, v_1) * (u_2, v_2) = (u_1 + v_1 u_2, v_1 v_2).$$

Thus the composition of $T^{u_1} \cdot v_1$ and $T^{u_2} \cdot v_2$ can be expressed in the following form

$$T^{u_1} \cdot v_1 \circ T^{u_2} \cdot v_2 = T^{u_1 + v_1 u_2} \cdot v_1 v_2.$$

The subgroup $\overrightarrow{\text{SL}}(\mathbb{Z}_{2k})$ of $\overrightarrow{\text{GL}}(\mathbb{Z}_{2k})$, consisting of the elements $T^u \cdot \pm 1$, is very familiar to musicians, for it consists of the (musical) transpositions and inversions, or combinations of both. The action of the supergroup $\overrightarrow{\text{GL}}(\mathbb{Z}_{2k})$ perhaps is not as intuitive, unless we abandon a one-dimensional view of the intervals and we visualize them over the surface of a torus, at least for the classical case of \mathbb{Z}_{12} .

This is possible for the following reason: The main theorem for finitely generated abelian groups (see the mathematical appendix, Section A.3) implies that we have an isomorphism

$$\tau : \mathbb{Z}_{12} \rightarrow \mathbb{Z}_3 \times \mathbb{Z}_4$$

because of the prime decomposition $12 = 3 \cdot 2^2$. In particular, we choose the isomorphism $x \mapsto \tau(x) = (x \bmod 3, -x \bmod 4)$, whose inverse is given by $(x_3, x_4) \mapsto 4x_3 + 3x_4$. Under τ , the major third is mapped to $(1, 0)$, while the minor third is sent to $(0, 1)$. The graph with $\mathbb{Z}_3 \times \mathbb{Z}_4$ as vertices and edges joining any two vertices that differ in exactly one minor third or one major third is a toroidal¹ graph, which we call the *torus of thirds* (Figure 2.1).

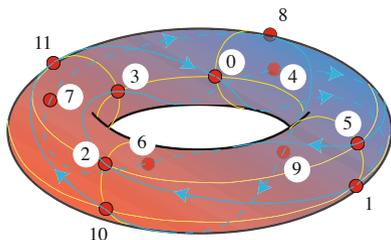


Fig. 2.1 The torus of thirds.

The periodicity of the octave can be readily captured by the group of complex numbers of unit module (the circle S^1) or the clock arithmetic. Since the underlying set of a torus is $S^1 \times S^1$, now we can see a double periodicity of the intervals: One cycle goes in steps of minor thirds, the other in steps of major thirds. With this picture in mind, we may visualize the multiplication by 5 as a reflection along the equatorial plane of the torus; by 7 as a reflection along a plane perpendicular to the toroidal plane; by 11, as a combination of the previous two. Translations can be seen to act as rotations in the toroidal or poloidal directions.

The actions of the complement and $\overrightarrow{\text{GL}}(\mathbb{Z}_{2k})$ can be considered simultaneously since they commute. This means we have an action of the direct product $\mathbb{Z}_2 \times \overrightarrow{\text{GL}}(\mathbb{Z}_{2k})$ on $\varepsilon.\mathbb{Z}_{2k}$. The orbits of this joint action are called *dichotomy classes*.

¹ This means that the graph can be embedded in the surface of a torus.

Definition 2.3. A marked dichotomy of intervals X is autocomplementary if its complement belongs to its orbit or, in other words, if its dichotomy class coincides with its marked dichotomy class. The dichotomy X is said to be rigid if the only symmetry that leaves it invariant is the identity, and if it is both autocomplementary and rigid it is called strong.

If X is a member of a strong dichotomy class and p is such that $p(X) = X^c$, then p is unique. Let q be another symmetry such that $q(X) = X^c$. Then

$$p^{-1}(q(X)) = p^{-1}(X^c) = X$$

which means that $p^{-1}q = T^0 \cdot 1$, where $T^0 \cdot 1$ is the identity of $\overrightarrow{\text{GL}}(\mathbb{Z}_{2k})$, and thus $p = q$ pre-multiplying both sides by p . Note also that from this equality we conclude that p is an involution, i.e., $p^2 = T^0 \cdot 1$. Being unique, p is called the *polarity* of (X/X^c) .

Example 2.2. In \mathbb{Z}_{12} , the marked dichotomy $X_1 = (\{0, 1, 3, 6, 8, 11\}/\{2, 4, 5, 7, 9, 10\})$ is not autocomplementary. On the other hand, $X_2 = (\{0, 1, 3, 7, 8, 11\}/\{2, 4, 5, 6, 9, 10\})$ is autocomplementary, because

$$\begin{aligned} T^9 \cdot 7(X_2) &= \{7 \cdot 0 + 9, 7 \cdot 1 + 9, 7 \cdot 3 + 9, 7 \cdot 7 + 9, 7 \cdot 8 + 9, 7 \cdot 11 + 9\} \\ &= \{9, 4, 6, 10, 5, 2\} = X_2^c. \end{aligned}$$

The set X_2 , nevertheless, is not strong because the symmetry $T^8 \cdot 5$ leaves it invariant. The dichotomy $K = \{0, 3, 4, 7, 8, 9\}$ is strong, because

$$\begin{aligned} T^2 \cdot 5(K) &= \{5 \cdot 0 + 2, 5 \cdot 3 + 2, 5 \cdot 4 + 2, 5 \cdot 7 + 2, 5 \cdot 8 + 2, 5 \cdot 9 + 2\} \\ &= \{2, 5, 10, 1, 6, 11\} = K^c \end{aligned}$$

and it is laborious, but direct, to check that apart from $T^0 \cdot 1$ no other symmetry leaves K invariant. Thus $p = T^2 \cdot 5$ is the polarity of K . This last example is of capital importance for this book, since K is the set of intervals from the tonic (within an octave) of the classical Renaissance consonances: prime (or unison), minor third, major third, perfect fifth, minor sixth and major sixth. The rest of the intervals (again, within an octave) are (of course) dissonances and are related to the consonances, via the polarity, in the following respective order: major second, perfect fourth, minor seventh, minor second, tritone, major seventh.

Problem 2.1. Verify in detail the claims of Example 2.2 (by the use of a computer, if possible).

It is not obvious that autocomplementary, rigid, or strong dichotomies exist in \mathbb{Z}_{2k} . But, whenever we have an involutory element p of $\overrightarrow{\text{GL}}(\mathbb{Z}_{2k})$ without fixed points, we can easily construct an autocomplementary dichotomy. Indeed, we first pick an arbitrary element $x_1 \in \mathbb{Z}_{2k}$, next we choose $x_2 \in \mathbb{Z}_{2k}$ distinct from x_1 and $p(x_1)$. Then we take x_j different from the elements

$$x_1, \dots, x_{j-1}, p(x_1), \dots, p(x_{j-1})$$

until we reach $j = k$. Thus $X = (\{x_1, \dots, x_k\}/\{p(x_1), \dots, p(x_k)\})$ is an autocomplementary marked dichotomy.

Lemma 2.1. *For any integer k and $0 \leq s \leq 2k$ odd, the automorphism $T^s \cdot -1 \in \overrightarrow{\text{GL}}(\mathbb{Z}_{2k})$ is involutory and has no fixed points.*

Proof. Since

$$(T^s \cdot -1) \circ (T^s \cdot -1) = T^s \circ T^{-s} \cdot 1 = T^0 \cdot 1 = 1$$

the involutoryity is settled. If $T^s \cdot -1(x) = s - x = x$ for some x , then $2x - s = 0$. This means that $2x - s$ is divisible by $2k$, which contradicts that $2x - s$ is odd. Consequently, $T^s \cdot -1$ does not have fixed points. \square

For \mathbb{Z}_2 it is obvious that $(\{1\}/\{0\})$ and $(\{0\}/\{1\})$ are its only strong marked interval dichotomies. For \mathbb{Z}_4 , there are no strong dichotomies. It suffices to check this for

$$X_1 = (\{0, 1\}/\{2, 3\}), X_2 = (\{0, 2\}/\{1, 3\}), X_3 = (\{0, 3\}/\{1, 2\}),$$

since all of them are autocomplementary. Nevertheless, $T^1 \cdot -1(X_1) = X_1$, $-1(X_2) = X_2$ and $T^{-1} \cdot -1(X_3) = X_3$, so none of them are rigid.

Now we can show that there exists at least one strong interval dichotomy in \mathbb{Z}_{2k} for $k \geq 3$. We should keep in mind that the invertible elements of \mathbb{Z}_{2k} are odd because they are coprime with $2k$.

Proposition 2.1. *Let $k \geq 3$. The dichotomy*

$$(X/Y) = (\{-1, 2k-2, 2k-4, \dots, 4, 2\}/\{0, 1, 3, \dots, 2k-5, 2k-3\})$$

in \mathbb{Z}_{2k} (which is obtained from the automorphism $T^{-1} \cdot -1$) is strong.

Proof. In general, the proposed dichotomy is clearly autocomplementary, with isomorphism $T^{-1} \cdot -1$. In order to prove its rigidity, we will show that for any automorphism $T^u \cdot w$, except the identity, at least one element of the marked interval dichotomy is mapped to an element in the complement. If $u = 0$, then $w(-1) = -w \neq -1$ is odd and therefore $w(-1) \in X^c$. If $u \neq 0$ is even, then $X \ni -w^{-1}u \neq 0$ is even and $T^u \cdot w(-w^{-1}u) = u - u = 0 \in X^c$. If u is odd, $T^u \cdot w(2) = u + 2w$ is odd. If it belongs to X^c , we are done. Otherwise, $T^u \cdot w(2) = u + 2w = -1$. It is impossible that $T^u \cdot w(4) = u + w = -1$, for it would imply that $2w = 0$ and $2 = 0$, contradicting that $k \geq 3$. \square

It is worthwhile to emphasize that strong dichotomies exist for any even cardinality except 4. This means that, in terms of this model, we cannot formulate a counterpoint theory for the 4-tone equally tempered scale. This scale can be interpreted as the tones of a diminished 7th chord in a twelve-tone equally tempered scale.

Remark 2.1. *The dichotomy*

$$((11, 10, 8, 6, 4, 2)/(0, 1, 3, 5, 7, 9)),$$

constructed for $k = 6$ in Proposition 2.1, belongs to the dichotomy class of

$$\Delta_{78} = (\{0, 1, 2, 4, 6, 10\}/\{3, 5, 7, 8, 9, 11\})$$

in Mazzola's list [57, appendix L].

2.2 Counterpoint Dichotomies

Now we need to extend the module of intervals so we can capture the idea of cantus firmus and discantus. In order to do so, we consider the ring of dual numbers

$$\mathbb{Z}_{2k}[\varepsilon] = \{a + \varepsilon.b : a, b \in \mathbb{Z}_{2k}, \varepsilon^2 = 0\},$$

and two dual numbers $a + \varepsilon.b, c + \varepsilon.d$ are equal if and only if $a = c$ and $b = d$. We define the sum and multiplication of dual numbers by

$$\begin{aligned} (a + \varepsilon.b) + (c + \varepsilon.d) &= (a + c) + \varepsilon.(b + d), \\ (a + \varepsilon.b)(c + \varepsilon.d) &= ac + \varepsilon.ad + \varepsilon.bc + \varepsilon^2.bd \\ &= ac + \varepsilon.(ad + bc). \end{aligned}$$

Within the dual number interpretation of counterpoint, a *contrapuntal interval* is a dual number $b + \varepsilon.b$, where the first component b , represents the cantus firmus, while the second one, b , represents the distance between the cantus firmus and the discantus. This justifies our election of the notation $\varepsilon.\mathbb{Z}_{2k}$ for the set of intervals: They are just the ε -components of contrapuntal intervals.

Example 2.3. In the twelve-tone scale, $2 + \varepsilon.7 \in \mathbb{Z}_{2k}[\varepsilon]$ represents a fifth over D. Thus the cantus firmus has the note D and the discantus has the note A, for $7 = 9 - 2$ or, equivalently, $9 = 2 + 7$. Observe that this interpretation of the discantus as being the sum of the two components of the interval $2 + \varepsilon.7$ defines what is called the “sweeping” counterpoint in literature. For the hanging interpretation, namely that the discantus would be $2 - 7 = 2 + 5 = 7$, we refer to Section 2.2.1 where the relations between sweeping and hanging counterpoint are discussed.

As with \mathbb{Z}_{2k} , we have the general affine group

$$\overrightarrow{\text{GL}}(\mathbb{Z}_{2k}[\varepsilon]) = \{T^{a+\varepsilon.b} \cdot (c + \varepsilon.d) : c + \varepsilon.d \in \mathbb{Z}_{2k}[\varepsilon]^\times\}$$

that acts on $\mathbb{Z}_{2k}[\varepsilon]$. Note that the set $\mathbb{Z}_{2k}^\times[\varepsilon]$ of invertible dual numbers consists of the dual numbers $a + \varepsilon.b$ with $a \in \mathbb{Z}_{2k}^\times$.

Since $\mathbb{Z}_{2k}[\varepsilon]$ has an even number of elements, dichotomies make sense for this ring, and thus we can talk about *marked counterpoint dichotomies* and *counterpoint dichotomy classes*.

Example 2.4. To each marked interval dichotomy (X/Y) , we can associate a marked counterpoint dichotomy $(X[\varepsilon]/Y[\varepsilon])$ called the *induced counterpoint dichotomy* by X , where

$$X[\varepsilon] = \{a + \varepsilon.b : a \in \mathbb{Z}_{2k}, b \in X\} := \mathbb{Z}_{2k} + \varepsilon.X.$$

Indeed

$$\begin{aligned} X[\varepsilon]^c &= \{a + \varepsilon.b : a \in \mathbb{Z}_{2k}, b \notin X\} \\ &= \{a + \varepsilon.b : a \in \mathbb{Z}_{2k}, b \in X^c = Y\} \\ &= \mathbb{Z}_{2k} + \varepsilon.Y = Y[\varepsilon]. \end{aligned}$$

If (X/Y) is a marked strong dichotomy, the induced counterpoint dichotomy $(X[\varepsilon]/Y[\varepsilon])$ is autocomplementary but not necessarily strong.

Problem 2.2. Provide an example of a marked strong dichotomy such that $X[\varepsilon]$ is not strong.

Problem 2.3. Check that if $T^u \cdot v$ is an autocomplementary symmetry of X , then $T^{\varepsilon.u} \cdot v$ is an autocomplementary symmetry of the induced counterpoint dichotomy $X[\varepsilon]$.

The following proposition will prove to be fundamental for the computational aspects of counterpoint.

Proposition 2.2. Let X be a marked strong dichotomy with polarity $p = T^u \cdot v$, and let $x \in \mathbb{Z}_{2k}$ be a cantus firmus. There exists exactly one symmetry $p^x[\varepsilon] \in \overrightarrow{\text{GL}}(\mathbb{Z}_{2k})$ (which will be called the induced polarity of $X[\varepsilon]$) such that

1. it is an autocomplementary function of $X[\varepsilon]$,
2. leaves $x + \varepsilon.\mathbb{Z}_{2k}$ (the set of all the counterpoint intervals with cantus firmus x) invariant.

It is given by

$$p^x[\varepsilon] = T^{x(1-v)+\varepsilon.u} \cdot v \tag{2.1}$$

and satisfies the following translational formula

$$p^{x+y}[\varepsilon] = T^x \circ p^y[\varepsilon] \circ T^{-x}. \tag{2.2}$$

Proof. It is straightforward to check that (2.1) satisfies the three requirements, so we only need to prove its unicity. Let $z \in X$. We are looking for some $u_1 + \varepsilon.v_1$ and $u_2 + \varepsilon.v_2$ such that the symmetry

$$p^x[\varepsilon] = T^{u_1+\varepsilon.v_1} \cdot (u_2 + \varepsilon.v_2)$$

satisfies

$$p^x[\varepsilon](x + \varepsilon.z) = x + \varepsilon.w.$$

This last equality states that $p^x[\varepsilon]$ sends an interval with cantus firmus x to another interval with the same cantus firmus. If we perform the calculations

$$\begin{aligned} p^x[\varepsilon](x + \varepsilon.z) &= T^{u_1 + \varepsilon.v_1} \cdot (u_2 + \varepsilon.v_2)(x + \varepsilon.z) \\ &= (u_1 + \varepsilon.v_1) + (u_2 + \varepsilon.v_2)(x + \varepsilon.z) \\ &= (u_1 + \varepsilon.v_1) + (u_2x + \varepsilon.(u_2z + v_2x)) \\ &= (u_1 + u_2x) + \varepsilon.(v_1 + u_2z + v_2x). \end{aligned}$$

then we deduce, by comparing the first components of the dual numbers, that we must have

$$u_1 + u_2x = x,$$

and thus

$$u_1 = x - u_2x = x(1 - u_2).$$

On the other hand, by the autocomplementarity, we must have $w \in Y$. This means that, for any t

$$v_1 + u_2z + v_2t = u_2z + (v_1 + v_2t) = T^{v_1 + v_2t} \cdot u_2(z) \in Y,$$

and the strength of X implies that $u_2 = v$ and $v_1 + v_2t = u$. Since t is arbitrary, the last equality holds for $t = 0$, so $v_1 = u$. Now, if $t = 1$, then $u + v_2 = u$, which implies $v_2 = 0$, and we are done. \square

2.2.1 Musical Meaning of the Operations with Counterpoint Intervals

Sums and multiplications of counterpoint intervals may seem of questionable musical significance. To prove that it is not so, let us exemplify the situation with $\mathbb{Z}_{12}[\varepsilon]$ and its group of affine symmetries

$$\overrightarrow{\text{GL}}(\mathbb{Z}_{12}[\varepsilon]) = \{T^{a+\varepsilon.b} \cdot (u + \varepsilon.v) : u = 1, 5, 7, 11\}.$$

Lemma 2.2. *Let $u \in \mathbb{Z}_{12}^\times$. The following identity holds*

$$(u + \varepsilon.v) = u(1 + \varepsilon.1)^s$$

for every $v \in \mathbb{Z}_{12}$, where s is any integer within the class of uv .

Proof. Since $u^2 = 1$ in \mathbb{Z}_{12} , then $u = u^{-1}$ and $(u + \varepsilon.v) = u(1 + \varepsilon.uv)$, so we only have to prove that $1 + \varepsilon.uv = (1 + \varepsilon)^s$ with $s \in \mathbb{Z}$ in the class of uv . In fact, $1 + \varepsilon.w = (1 + \varepsilon)^{s'}$, with s' in the class of w . For $w = 0$, we have the calculation $1 + \varepsilon.0 = 1 = (1 + \varepsilon.0)^0$.

If this is true for w , then

$$\begin{aligned}(1 + \varepsilon)^{s'+1} &= (1 + \varepsilon.1)^{s'}(1 + \varepsilon.1) \\ &= (1 + \varepsilon.w)(1 + \varepsilon.1) \\ &= 1 + \varepsilon.w + \varepsilon.1 = 1 + \varepsilon.(w + 1),\end{aligned}$$

and it is clear that $s' + 1$ belongs to the class of $w + 1$, and we are done by induction. \square

By Lemma 2.2, we may write

$$T^{a+\varepsilon.b} \cdot (u + \varepsilon.v) = T^a T^{\varepsilon.b} \cdot u(1 + \varepsilon.1)^s$$

for an element $\overrightarrow{\text{GL}}(\mathbb{Z}_{12}[\varepsilon])$. Thus, to understand the effects of interval multiplication, it suffices to examine the four symmetries

$$s_1 = T^a, s_2 = T^{\varepsilon.b}, s_3 = T^0 \cdot u, s_4 = T^0 \cdot (1 + \varepsilon.1).$$

1. For s_1 , the identity $T^a \cdot (x + \varepsilon.y) = (x + a) + \varepsilon.y$ expresses that the whole interval is transposed by a .
2. For s_2 , according to $T^{\varepsilon.b} \cdot (x + \varepsilon.y) = x + \varepsilon.(y + b)$, it causes the interval y to be transposed by b , leaving the cantus firmus unchanged. This kind of operation is common in double counterpoint.
3. When it comes to s_3 , we have to consider three cases apart from the identity. Perhaps $u = 11 = -1$ is the most natural because it inverts the interval; it also reflects the cantus firmus with respect the tonic, but it can be restored later via a translation T^x . This operation is also common in double counterpoint. The remaining cases $u = 5, 7$ are not as natural unless we resort again to the torus of thirds as described in Section 2.1.
4. The symmetry obtained from s_4 suggests an interesting transformation, since

$$(1 + \varepsilon.1)(a + \varepsilon.b) = a + \varepsilon.a + b,$$

which means that the discantus is transposed by the same interval between the tonic and the cantus firmus, but without moving the cantus firmus itself. This operation can be iterated, generating a cycle of counterpoint intervals that return to the original one, because the underlying ring is cyclic with respect to addition.

Another reason for the importance of the symmetry $1 + \varepsilon.1$ is that it allows us to handle voice crossings. For a counterpoint interval $x + \varepsilon.y$, we may define an *orientation*, which can be hanging or sweeping, that indicates how to obtain the discantus tone. If it is sweeping, the discantus is $x + y$, while if it is hanging, it is $x - y$. A little more formally, we have the mappings

$$\begin{aligned}\alpha_+ : \mathbb{Z}_{12}[\varepsilon] &\longrightarrow \mathbb{Z}_{12}, \\ x + \varepsilon.y &\longmapsto x + y,\end{aligned}$$

and

$$\begin{aligned}\alpha_- : \mathbb{Z}_{12}[\varepsilon] &\longrightarrow \mathbb{Z}_{12}, \\ x + \varepsilon.y &\longmapsto x - y.\end{aligned}$$

Observe that $-\alpha_+ \circ T^0 \cdot (1 + \varepsilon.1)^{-2} = \alpha_-$, since

$$\begin{aligned}-\alpha_+((1 + \varepsilon.1)^{-2}(x + \varepsilon.y)) &= -\alpha_+((1 + \varepsilon.(-2))(x + \varepsilon.y)) \\ &= -\alpha_+(x + \varepsilon.y - 2x) \\ &= -(y - x) = x - y = \alpha_-(x + \varepsilon.y).\end{aligned}$$

Thus, if we have a hanging counterpoint interval $a + \varepsilon.b$, we may write

$$\begin{aligned}\alpha_-(a + \varepsilon.b) &= -\alpha_+((1 + \varepsilon.1)^{-2}(a + \varepsilon.b)) \\ &= \alpha_+(-(1 + \varepsilon.1)^{-2}(a + \varepsilon.b)),\end{aligned}$$

which means that the sweeping counterpoint interval

$$-(1 + \varepsilon.1)^{-2}(a + \varepsilon.b) = -(a + \varepsilon.b - 2a) = -a + \varepsilon.(2a - b)$$

can be regarded as equivalent to the original one.

2.3 Counterpoint Symmetries

We have arrived at the crux of the model. As we have discussed in Section 1.3, we can generate tension in the confrontation of a consonance against another by the *deformation* of consonances into dissonances and dissonances into consonances. We can now make this precise, saying that for a symmetry g , the marked interval dichotomy $(gX[\varepsilon]/gY[\varepsilon])$ is a *deformation* of the original dichotomy $(X[\varepsilon]/Y[\varepsilon])$. Thus, if we can interpret a consonant interval $\xi \in X[\varepsilon]$ as a deformed dissonance, i.e., $\xi \in gY[\varepsilon]$, we can “resolve” it in a deformed consonance that is also a consonance, i.e., a member η of $gX[\varepsilon] \cap X[\varepsilon]$.

Definition 2.4. Let $\{\xi, \eta\}$ be a pair of counterpoint intervals from the dichotomy $(X[\varepsilon]/Y[\varepsilon])$. If there exists at least one symmetry $g \in \overrightarrow{\text{GL}}(\mathbb{Z}_{2k}[\varepsilon])$ such that $\xi \in g.Y[\varepsilon]$ and $\eta \in g.X[\varepsilon]$, then we say that $\{\xi, \eta\}$ is g -polarized.

Theorem 2.1. Let $(X[\varepsilon]/Y[\varepsilon])$ a dichotomy and ξ and η two different intervals. Then there exists one symmetry $g \in \overrightarrow{\text{GL}}(\mathbb{Z}_{2k}[\varepsilon])$ such that $\{\xi, \eta\}$ is g -polarized.

Proof. If already ξ and η belong to the complementary sets defined by $(X[\varepsilon]/Y[\varepsilon])$, the identity $T^1 \cdot 1$ does the job. Suppose first that $\xi = u_1 + \varepsilon.v_1$ and $\eta = u_2 + \varepsilon.v_2$ with $v_1 \neq v_2$. We claim that there exists m such that the symmetry $T^{\varepsilon.m} \cdot 1$ polarizes the pair. If it were not the case, then for any m we would have

$$v_1 + m, v_2 + m \in X.$$

In particular, if we choose an arbitrary a and set $m = a - v_1$, then $a, v_2 - v_1 + a \in X$. Thus $T^{v_2 - v_1} \cdot 1$ leaves $X[\varepsilon]$ invariant, which forces $v_2 - v_1 = 0$ and $v_2 = v_1$, a contradiction.

It remains to examine the case $u_1 \neq u_2$. Now we claim that there exists an m such that $T^{\varepsilon \cdot m} \cdot (1 + \varepsilon \cdot 1)$ polarizes the pair. Otherwise, $v_1 + u_1 + m, v_2 + u_2 + m \in X$, and we may choose an arbitrary a to define $m = a - u_1 - v_2$, rendering $T^{u_2 - u_1} \cdot 1$ an automorphism of X . Thus $u_2 = u_1$ and we have a contradiction again. This completes the proof. \square

Definition 2.5. Let $(X[\varepsilon]/Y[\varepsilon])$ be a dichotomy and $\xi \in X[\varepsilon]$ a counterpoint interval. A symmetry $g \in \overrightarrow{\text{GL}}(\mathbb{Z}_{2k}[\varepsilon])$ is contrapuntal if

1. the interval ξ does not belong to $g(X[\varepsilon])$,
2. the symmetry $p^x[\varepsilon]$ is a polarity of $(g(X[\varepsilon])/g(Y[\varepsilon]))$,
3. the cardinality of $g(X[\varepsilon]) \cap X[\varepsilon]$ is maximal among the symmetries with the previous two properties.

Definition 2.6. Given a dichotomy $(X[\varepsilon]/Y[\varepsilon])$ and an interval $\xi \in X[\varepsilon]$, we say η is an admitted successor if ξ is contained in $g(X[\varepsilon])/X[\varepsilon]$ for some contrapuntal symmetry g .

Example 2.5. Consider the dichotomy

$$(K/D) = (\{0, 3, 4, 7, 8, 9\}/\{1, 2, 5, 6, 10, 11\})$$

in \mathbb{Z}_{12} from Example 2.2, the interval $\xi = \varepsilon \cdot 9$ (a major sixth over the tonic) and the symmetry $g = T^{\varepsilon \cdot 8} \cdot (5 + \varepsilon \cdot 4)$. The cantus firmus is $x = 0$, and by Proposition 2.2 we know that a polarity of $(K[\varepsilon]/D[\varepsilon])$ is $p^0[\varepsilon] = T^{\varepsilon \cdot 2} \cdot 5$. Let us verify that g is a counterpoint symmetry for ξ . First,

$$\begin{aligned} g(K[\varepsilon]) &= (1 - \varepsilon \cdot 4)\mathbb{Z}_{2k} + \varepsilon \cdot T^8 \cdot 5K \\ &= (1 - \varepsilon \cdot 4)\mathbb{Z}_{2k} + \varepsilon \cdot T^6 T^2 \cdot 5K \\ &= (1 - \varepsilon \cdot 4)\mathbb{Z}_{2k} + \varepsilon \cdot T^6 D, \end{aligned}$$

which means that the elements of $gK[\varepsilon]$ are of the form

$$w + \varepsilon \cdot T^{6-4w} \cdot y$$

for some $y \in D$. This means $\xi \notin gK[\varepsilon]$, since $9 \notin T^6 D = \{0, 4, 5, 7, 8, 11\}$. Second,

$$\begin{aligned} p^0[\varepsilon](gK[\varepsilon]) &= T^{\varepsilon \cdot 2} \cdot 5((1 + \varepsilon \cdot 4)\mathbb{Z}_{2k} \varepsilon \cdot T^6 D) \\ &= (1 - \varepsilon \cdot 4)5\mathbb{Z}_{2k} + \varepsilon \cdot T^8 \cdot 5K \\ &= (1 - \varepsilon \cdot 4)\mathbb{Z}_{2k} + \varepsilon \cdot T^6 T^2 \cdot 5K \\ &= (1 - \varepsilon \cdot 4)\mathbb{Z}_{2k} + \varepsilon \cdot T^6 K = gD[\varepsilon] \end{aligned}$$

which precisely means that $p^0[\varepsilon]$ is a polarity of $(K[\varepsilon]/D[\varepsilon])$. It is not difficult to calculate that

$$|(gK[\varepsilon]) \cap K[\varepsilon]| = 56$$

and that this is the maximum possible. Thus, g is a counterpoint symmetry.

Take again the dichotomy (K/D) . The consonances with cantus firmus w are

$$K_w := w + \varepsilon.K,$$

and the action of $p^0[\varepsilon]$ over them is given by

$$p^0[\varepsilon](K_w) = T^{\varepsilon \cdot 2} \cdot 5(w + \varepsilon.K) = 5w + \varepsilon.T^2 \cdot 5K = 5w + \varepsilon.D,$$

which means that this polarity of counterpoint intervals is equivalent to applying the translation T^{4w} to the cantus firmus, and the marked interval polarity to the interval. In this sense, $p^0[\varepsilon]$ is global, since for any consonances with fixed cantus firmus, w acts in the same manner.

This is different for “deformed” consonances by the symmetry g with cantus firmus w :

$$gK_w := (w\varepsilon.\mathbb{Z}_{2k}) \cap gK[\varepsilon]$$

For instance, if g is the counterpoint symmetry of the last example, we have

$$gK_w = w + \varepsilon.T^{8-4w} \cdot 5K$$

and $p^0[\varepsilon]$ does not act by translating gK_w with T^{4w} and applying the polarity to the interval. Instead

$$p^0[\varepsilon](gK_w) = (5w + \varepsilon.\mathbb{Z}_{2k}) \cap (5w + \varepsilon.T^{6-4w}K),$$

thus the way $p^0[\varepsilon]$ “distorts” the consonances depends on the value of the cantus firmus. In this sense, $p^0[\varepsilon]$ is a *local* symmetry for the deformed dichotomy $(gX[\varepsilon]/gY[\varepsilon])$.

2.4 The Counterpoint Theorem

In terms of raw computational power, from this point on we can simply take any interval in a counterpoint dichotomy and test all the available transformations, looking for its counterpoint symmetries, and then we can make a list of its admitted successors to realize the counterpoint theory in an equally tempered \mathbb{Z}_{2k} -tone scale. Nevertheless, it is important to study the matter more carefully in order to optimize this naive algorithm and derive further properties of the model.

2.4.1 Some Preliminary Calculations

Lemma 2.3. *Let (X/Y) be a strong dichotomy and $(X[\varepsilon]/Y[\varepsilon])$ its induced counterpoint dichotomy. The symmetries that leave $X[\varepsilon]$ invariant are of the form*

$$T^{\mathbb{Z}_{2k}} := \{T^w : w \in \mathbb{Z}_{2k}\}.$$

Proof. Every element of $T^{\mathbb{Z}_{2k}}$ is a symmetry of $X[\varepsilon]$, since

$$T^w(y + \varepsilon.x) = (x + w) + \varepsilon.x \in X[\varepsilon].$$

With this in mind, it is clear that $T^{z+\varepsilon.w} \cdot (u + \varepsilon.v)$ is a symmetry if and only if $T^{\varepsilon.w} \cdot (u + \varepsilon.v)$ is a symmetry. In particular, we would need that

$$w + vy + ux \in X$$

for every $y \in \mathbb{Z}_{2k}$ and $x \in X$. Since X is strong, we have $u = 1$ and $w + vy = 0$. In particular, for $y = 0$, we have $x + w \in X$ for every w , and strength implies that $w = 0$. Thus $vx = 0$, which is also valid when $y = 1$, and therefore $v = 0$. \square

For the following, we define the group

$$H := T^{\varepsilon.\mathbb{Z}_{2k}} \text{GL}(\mathbb{Z}_{2k}[\varepsilon]) = \{T^{\varepsilon.t} \cdot (u + \varepsilon.v) : u + \varepsilon.v \in \mathbb{Z}_{2k}^\times\}.$$

Lemma 2.4. *For $g = T^{\varepsilon.t} \cdot (u + \varepsilon.v)$ and $z \in \mathbb{Z}_{2k}$, define*

$$g^{(z)} = gT^{\varepsilon.u^{-2}vz} \in H.$$

Then

$$(g^{(z_1)})^{(z_2)} = g^{(z_1+z_2)} \tag{2.3}$$

and

$$T^z gX[\varepsilon] = g^{(-z)}X[\varepsilon]. \tag{2.4}$$

Proof. The first equality is direct:

$$\begin{aligned} (g^{(z_1)})^{(z_2)} &= (g^{(z_1)})T^{\varepsilon.u^{-2}vz_2} \\ &= (gT^{\varepsilon.u^{-2}vz_1})T^{\varepsilon.u^{-2}vz_2} \\ &= g(T^{\varepsilon.u^{-2}vz_1}T^{\varepsilon.u^{-2}vz_2}) \\ &= gT^{\varepsilon.u^{-2}v(z_1+z_2)} \\ &= g^{(z_1+z_2)}. \end{aligned}$$

The second equality is a little more involved. First

$$\begin{aligned} T^z g &= T^z \cdot (u + \varepsilon.v) = T^{\varepsilon.t} \cdot (u + \varepsilon.v) T^{z(u^{-1} - \varepsilon.u^{-2}v)} \\ &= g^{(-z)} T^{zu^{-1}} \end{aligned}$$

and thus

$$T^z gX[\varepsilon] = g^{(-z)} T^{zu^{-1}} X[\varepsilon] = g^{(-z)} X[\varepsilon]$$

using Lemma 2.3. □

Corollary 2.1. *For every $g \in \overrightarrow{\text{GL}}(\mathbb{Z}_{2k}[\varepsilon])$, there exists a symmetry $h \in H$ such that $gX[\varepsilon] = hX[\varepsilon]$.*

Proof. Given $g = T^{u_1 + \varepsilon.v_1} \cdot (u_2 + \varepsilon.v_2)$, define

$$f = T^{\varepsilon.v_1} \cdot (u_2 + \varepsilon.v_2).$$

Then $g = T^{u_1} f$, and by Lemma 2.4

$$gX[\varepsilon] = T^{u_1} fX[\varepsilon] = f^{(-z)} X[\varepsilon],$$

hence $h = f^{(-z)}$ is the required symmetry. □

Lemma 2.5. *Let $\xi = x + \varepsilon.k$, $g \in \overrightarrow{\text{GL}}(\mathbb{Z}_{2k}[\varepsilon])$ and $y \in \mathbb{Z}_{2k}$. If*

$$\xi \notin gX[\varepsilon] \quad \text{and} \quad p^x[\varepsilon] : gX[\varepsilon] \rightarrow gY[\varepsilon]$$

where $p^x[\varepsilon]$ is the induced polarity of $X[\varepsilon]$, then

$$T^z \xi \notin T^z gX[\varepsilon] \quad \text{and} \quad p^{z+x}[\varepsilon] : T^z gX[\varepsilon] \rightarrow T^z gY[\varepsilon].$$

Furthermore,

$$(T^z gX[\varepsilon]) \cap X[\varepsilon] = T^z \cdot (gX[\varepsilon] \cap X[\varepsilon]).$$

Proof. It is clear that $T^z \xi \notin T^z gX[\varepsilon]$, since $T^z \in T^{\mathbb{Z}_{2k}}$ is bijective. Using the translation formula (2.2), we can write

$$\begin{aligned} p^{z+x}[\varepsilon] T^z gX[\varepsilon] &= p^{x+z}[\varepsilon] T^z gX[\varepsilon] \\ &= T^z p^x[\varepsilon] T^{-z} T^z gX[\varepsilon] \\ &= T^z p^x[\varepsilon] gX[\varepsilon] \\ &= T^z gY[\varepsilon]. \end{aligned}$$

For the last part of this lemma, we invoke Lemma 2.3,

$$(T^z g[\varepsilon]) \cap X[\varepsilon] = (T^z g[\varepsilon]) \cap (T^z X[\varepsilon]) = T^z (gX[\varepsilon] \cap X[\varepsilon]),$$

since $g(\eta_1) = \eta_2 = T^{-z} T^z \eta_2$ for some η_1 and η_2 if and only if $T^z \cdot (g(\eta_1)) = T^z \cdot (\eta_2)$. □

Now we have all the necessary ingredients to prove a key theorem simplifying the algorithm for the calculation of contrapuntal symmetries; it says the intervals with respect to the tonic are the essential ones.

Theorem 2.2. *If $\xi = x + \varepsilon.k \in X[\varepsilon]$ is a consonant interval and g is one of its counterpoint symmetries, then there is a symmetry $h \in H$ such that we can verify it is contrapuntal using the set $hX[\varepsilon] = gX[\varepsilon]$. Furthermore, to check the required properties, we can restrict ourselves to the interval $\varepsilon.k$, the symmetry $h^{(x)} \in H$, and the polarity $p^0[\varepsilon]$. Last, but not least, the set of admitted successors $(hX[\varepsilon]) \cap X[\varepsilon]$ coincides with the set*

$$T^x \cdot (h^{(x)}X[\varepsilon]) \cap X[\varepsilon].$$

In sum: to calculate the admitted successors of $x + \varepsilon.k$, we may first calculate those of $0 + \varepsilon.k$ with the smaller group of candidates H and translate the results accordingly once finished.

Proof. The first replacement of g is justified by Lemma 2.1. For the second replacement, by Lemma 2.4 we have

$$T^{-x}hX[\varepsilon] = h^{(x)}X[\varepsilon]$$

and using Lemma 2.5 with $z = -x$, we may verify that h is contrapuntal examining $h^{(x)}$ with the interval $T^{-x}\xi = \varepsilon.k$ and the polarity $p^{-x+x}[\varepsilon] = p^0[\varepsilon]$. From Lemma 2.5 we also have

$$h^{(x)}X[\varepsilon] \cap X[\varepsilon] = T^{-x}hX[\varepsilon] \cap X[\varepsilon] = T^{-x}(hX[\varepsilon] \cap X[\varepsilon])$$

and thus

$$hX[\varepsilon] \cap X[\varepsilon] = T^x \cdot ((h^{(x)}X[\varepsilon]) \cap X[\varepsilon])$$

as claimed. □

The next lemma from additive combinatorics is useful to prove a weak version of the so-called *counterpoint theorem*.

Lemma 2.6. *Let S be a finite subset of a cyclic group \mathbb{Z}_n and $u \in \mathbb{Z}_n^\times$. Then*

$$\sum_{x \in \mathbb{Z}_n} |T^x \cdot u(S) \cap S| = |S|^2.$$

Proof. This is a straightforward consequence of the associativity and commutativity of the sum:

$$\begin{aligned}
\sum_{x \in \mathbb{Z}_n} |T^x \cdot u(S) \cap S| &= \sum_{x \in \mathbb{Z}_n} \sum_{y \in S} \sum_{z \in S} [uz + x = y] \\
&= \sum_{y \in S} \sum_{z \in S} \sum_{x \in \mathbb{Z}_n} [uz + x = y] \\
&= \sum_{y \in S} \sum_{z \in S} \sum_{x' \in \mathbb{Z}_n} [x' = y] \\
&= \sum_{y \in S} \sum_{z \in S} 1 \\
&= \sum_{y \in S} |S| \\
&= |S|^2.
\end{aligned}$$

The third equality follows from the fact that \mathbb{Z}_n being a cyclic group with respect to addition, it is generated by any of its elements. \square

2.4.2 Hichert's Algorithm

We are ready now to examine the final details regarding Hichert's algorithm for a more efficient calculation of contrapuntal symmetries. Let $\Delta = (X/Y)$ be a strong dichotomy, $\xi = \varepsilon.k$ with $k \in X$, and the symmetry $g = T^{\varepsilon.t} \cdot (u + \varepsilon.uv) \in H$; without loss of generality, we may choose uv in the linear part of the symmetry because u is invertible. Let us restate the conditions for contrapuntal character of these settings. To begin with, we have

$$\begin{aligned}
gX[\varepsilon] &= \bigcup_{x \in \mathbb{Z}_{2k}} g(x + \varepsilon.X) \\
&= \bigcup_{x \in \mathbb{Z}_{2k}} (ux + \varepsilon.(uvx + t) + \varepsilon.uX) \\
&= \bigcup_{y \in \mathbb{Z}_{2k}} (y + \varepsilon.(vy + t) + \varepsilon.uX) = \bigcup_{y \in \mathbb{Z}_{2k}} (y + \varepsilon.T^{vy+t}u(X)).
\end{aligned}$$

Letting $f(y) = T^{vy+t}u$, we rewrite the latter as

$$gX[\varepsilon] = \bigcup_{y \in \mathbb{Z}_{2k}} (y + \varepsilon.f(y)(X)) \quad (2.5)$$

and

$$(gX[\varepsilon]) \cap X[\varepsilon] = \bigcup_{y \in \mathbb{Z}_{2k}} (y + \varepsilon.(f(y)(X) \cap X)). \quad (2.6)$$

Thus, from (2.5) it follows that the first condition for contrapuntality reduces to $k \notin f(0)X$. In other words, $k \in f(0)Y = f(0)p(X)$, which means that

$$k = f(0)p(s) = t + up(s)$$

for some $s \in X$. Hence

$$t \in \{k - up(s) : s \in X\}. \quad (2.7)$$

And thus there exists $s \in X$ such that

$$gX[\varepsilon] = \bigcup_{y \in \mathbb{Z}_{2k}} (y + \varepsilon \cdot (T^{vy+k-up(s)} \cdot u(X)))$$

and therefore

$$|(gX[\varepsilon]) \cap X[\varepsilon]| = \sum_{y \in \mathbb{Z}_{2k}} |T^{vy+k-up(s)} \cdot u(X) \cap X|. \quad (2.8)$$

This expression for the cardinality of the intersection of deformed consonances and consonances is of great help for the estimation of its size.

Lemma 2.7. *Let $g = T^{\varepsilon \cdot t} \cdot (u + \varepsilon \cdot v) \in H$. Then the fact that $p^0[\varepsilon]$ is a polarity of $gX[\varepsilon]$*

$$p^0[\varepsilon]gX[\varepsilon] = gp^0[\varepsilon]X[\varepsilon] = gY[\varepsilon]$$

is equivalent to the commutativity condition

$$p^0[\varepsilon]g = gp^0[\varepsilon].$$

Proof. The sufficiency is obvious. For the necessity, let $\alpha + \varepsilon \cdot \beta$ be any interval with $\beta \in X$. By hypothesis, there exists a $\gamma + \varepsilon \cdot \delta$ with $\delta \in X$ such that

$$p^0[\varepsilon]g(\alpha + \varepsilon \cdot \beta) = gp^0[\varepsilon](\gamma + \varepsilon \cdot \delta).$$

Writing $p = T^r w$, we calculate

$$\begin{aligned} p^0[\varepsilon]g(\alpha + \varepsilon \cdot \beta) &= T^{\varepsilon \cdot r} \cdot w(u\alpha + \varepsilon \cdot (v\alpha + u\beta + t)) \\ &= uw\alpha + \varepsilon \cdot (vw\alpha + uw\beta + wt + r) \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} gp^0[\varepsilon](\gamma + \varepsilon \cdot \delta) &= T^{\varepsilon \cdot t} (u + \varepsilon \cdot v)(w\gamma + \varepsilon \cdot (w\delta + r)) \\ &= uw\gamma + \varepsilon \cdot (uw\delta + vw\gamma + ur + t). \end{aligned}$$

We are led to the system

$$\begin{aligned} uw\alpha &= uw\gamma, \\ vw\alpha + uw\beta + wt + r &= uw\delta + vw\gamma + ur + t. \end{aligned} \quad (2.10)$$

That allows us to deduce, using the invertibility of u and w , that

$$\begin{aligned}\gamma &= \alpha, \\ \delta &= \beta + ut(1-w) + wr(u-1) \in X.\end{aligned}$$

Since Δ is strong, we must have $ut(1-w) + wr(u-1) = 0$, since the second equality holds for an arbitrary $\beta \in X$. Hence $\delta = \beta$ and $p^0[\varepsilon]g = gp^0[\varepsilon]$. \square

Remark 2.2. *It is important to stress that the previous lemma is false if $g \notin H$. For example, take $g = T^1 \cdot 1$ and the induce polarity $p^0_{(K/D)}[\varepsilon] = T^{\varepsilon \cdot 2} \cdot 5$ for the dichotomy (K/D) from Example 2.2. It is clear that $p^0_{(K/D)}[\varepsilon]$ is a polarity of $(gK[\varepsilon]/gD[\varepsilon]) = (K[\varepsilon]/D[\varepsilon])$. Nevertheless*

$$\begin{aligned}g \circ p^0_{(K/D)}[\varepsilon] &= T^1 \cdot 1 \circ T^{\varepsilon \cdot 2} \cdot 5 \\ &= T^{1+\varepsilon \cdot 2} \cdot 5 \\ &\neq T^{5+\varepsilon \cdot 2} \cdot 5 \\ &= T^{\varepsilon \cdot 2} \cdot 5 \circ T^1 \cdot 1 = p^0_{(K/D)}[\varepsilon] \circ g.\end{aligned}$$

This lemma yields a quick criterion to check if $gX[\varepsilon]$ is polarized by $p^0[\varepsilon]$. Note that from (2.10) we know that the condition

$$wt + r = ur + t \tag{2.11}$$

must hold, because $\alpha = \gamma$ and $\beta = \delta$.

To finish the description of Hichert's algorithm, we need to analyze more carefully (2.8). If $\rho = \gcd(v, 2k)$, then

$$|gX[\varepsilon] \cap X[\varepsilon]| = \rho \sum_{j=0}^{\frac{2k}{\rho}-1} |(T^{j\rho+k-up(s)} \cdot u(X)) \cap X|.$$

When $\rho = 1$, this reduces to

$$|gX[\varepsilon] \cap X[\varepsilon]| = \sum_{y \in \mathbb{Z}_{2k}} |T^y \cdot u \cdot X \cap X| = |X|^2 = k^2. \tag{2.12}$$

by Lemma 2.6.

Now observe that the cardinality of $T^{y-up(s)} \cdot uX \cap X$ cannot exceed $k-1$, because (X/Y) is strong and $y-p(s) \neq 0$, since p is the polarity. Therefore,

$$2k|T^{y-up(s)} \cdot u(X) \cap X| \leq 2k(k-1).$$

The equation $j\rho + y - p(s) = 0$ has at most one solution in the interval $0 \leq j < 2k/\rho$. This means that

$$\begin{aligned}
\rho \sum_{j=0}^{2k/\rho-1} |T^{j\rho+y-up(s)} \cdot u(X) \cap X| &\leq \rho \left[\left(\frac{2k}{\rho} - 1 \right) (k-1) + k \right] \\
&= \rho + 2k(k-1) \\
&\leq k + 2k(k-1) = 2k^2 - k.
\end{aligned}$$

Summarizing, we have the following result.

Theorem 2.3 (Kleiner Kontrapunktsatz, Mazzola [53], Agustín-Aquino [4]). *Let $\Delta = (X/Y)$ be a strong interval dichotomy, and let $\xi \in X[\varepsilon]$. The number N of admitted successors of ξ satisfies*

$$k^2 \leq N \leq 2k^2 - k.$$

Remark 2.3. *Both bounds are tight: The dichotomy*

$$W = (\{0, 1, 3\} / \{2, 4, 5\})$$

in \mathbb{Z}_6 is strong,² and the number of admitted successors of $\varepsilon.1$ is exactly $15 = 2 \cdot 3^2 - 3$. Its unique contrapuntal symmetry is $g = T^{\varepsilon.3} \cdot (1 + \varepsilon.3)$. On the other hand, the dichotomy

$$X = (\{1, 4, 5, 6, 7, 8, 14, 15\} / \{0, 2, 3, 9, 10, 11, 12, 13\})$$

in \mathbb{Z}_{16} is such that the number of admitted successors of $\varepsilon.6$ is $64 = 8^2$, and has the impressive number of 105 contrapuntal symmetries (considering that H in this case consist of 128 symmetries).

Example 2.6. The case of \mathbb{Z}_6 is an interesting illustration of the generalization from 12 to $2k$, because it represents the whole-tone scale. On the one hand, the restriction of the (K/D) dichotomy to whole-tone intervals yields the sets

$$\bar{K} = \{0, 4, 8\}, \quad \bar{D} = \{2, 6, 10\}.$$

The set \bar{K} contains the prime, the major third, and the minor sixth, while the major second, the tritone, and the minor seventh belong to \bar{D} . In the whole-tone scale, this corresponds to the dichotomy

$$(\{0, 2, 4\} / \{1, 3, 5\}),$$

which, unfortunately, is not strong ($T^3 \cdot 1$ is a non-trivial automorphism). On the other hand, the major second is “consonant” in the aforementioned dichotomy W , and the set of admitted successors for the interval $\varepsilon.1$ (ascending major second over \mathbb{C}), is

² It is essentially the *only* strong dichotomy in \mathbb{Z}_6 , except for affine isomorphic images.

$$S = gX[\varepsilon] \cap X[\varepsilon] = \{\varepsilon.0, \varepsilon.3, 1 + \varepsilon.0, 1 + \varepsilon.1, 1 + \varepsilon.3, \\ 2 + \varepsilon.0, 2 + \varepsilon.3, 3 + \varepsilon.0, 3 + \varepsilon.1, 3 + \varepsilon.3, \\ 4 + \varepsilon.0, 4 + \varepsilon.3, 5 + \varepsilon.0, 5 + \varepsilon.1, 5 + \varepsilon.3\},$$

with the claimed 15 elements. In fact, the only “forbidden” successors (apart from the interval itself) are $2 + \varepsilon.1$ (the major second over E) and $4 + \varepsilon.1$ (the major second over Ab). In other words, parallel major second progressions producing a “mi contra fa” cross relation are not allowed: C-D going to E-F \sharp and C-D going to Ab-Bb.

All of the components of Hichert’s algorithm are finally ready.

Algorithm 2.1 (J. Hichert, 1993, [38]). We calculate the contrapuntal symmetries within H for the intervals $\varepsilon.k \in X[\varepsilon]$. Here $\chi(x,y)$ is a function that outputs the cardinality of the set $(T^x \cdot y(X)) \cap X$.

Require: The strong dichotomy $\Delta = (X/Y)$ in \mathbb{Z}_{2n} and its polarity $T^r \cdot w$.

Ensure: The set of counterpoint symmetries $\Sigma_k \subseteq H$ for each $\varepsilon.k \in X[\varepsilon]$.

```

1  foreach  $k \in X$  {
2       $M \leftarrow 0, \Sigma_k \leftarrow \emptyset$ ;
3      foreach  $u \in \text{GL}(\mathbb{Z}_{2n})$  {
4          foreach  $s \in X$  {
5              foreach  $v \in \mathbb{Z}_{2n}$  {
6                   $t \leftarrow k - u(ws + r)$ ;
7                  if  $wt + r = ur + t$  {
8                      if  $v = 0$  {
9                           $S \leftarrow 2n\chi(t, u)$ ;
10                     }
11                    else if  $v \in \text{GL}(\mathbb{Z}_{2n})$  {
12                         $S \leftarrow n^2$ ;
13                    }
14                    else {
15                         $\rho \leftarrow \text{gcd}(v, 2n)$ ;
16                         $S \leftarrow \rho \sum_{j=0}^{\frac{2n}{\rho}-1} \chi(j\rho + t, u)$ ;
17                    }
18                    if  $S > M$  {
19                         $\Sigma_k \leftarrow \{T^{\varepsilon.t} \cdot (u + \varepsilon.uv)\}$ ;
20                         $M \leftarrow S$ ;
21                    }
22                    else if  $S = M$  {
23                         $\Sigma_k \leftarrow \Sigma_k \cup \{T^{\varepsilon.t} \cdot (u + \varepsilon.uv)\}$ ;
24                    }
25                }
26            }
27        }
28    }
29    return  $(\Sigma_k)$ ;
30 }
```

Proof. The election of t in line 6 is justified by (2.7). In line 7 we check that the symmetry polarizes (gX/gY) using (2.11). The lines from 8 to 17 calculate the cardinality of $gX \cap X$ according to (2.12). Afterwards, from line 18 to 21, we update the set of contrapuntal symmetries S_k in a standard way. Since there are a finite number of symmetries to analyze, the algorithm terminates and Σ_k contains all the contrapuntal symmetries within H . \square

Chapter 3

The Case of the Twelve-Tone Scale

The dichotomies in \mathbb{Z}_{12} are special, for they represent those of the standard twelve-tone scale. They have been completely determined, and we know there are 34 marked dichotomy classes, which define 26 dichotomy classes. Among these, exactly 8 are autocomplementary, and only 6 are strong. Two strong dichotomies are of particular notability: the *Ionian* or *major* dichotomy

$$(I/J) = (\{2, 4, 5, 7, 9, 11\} / \{0, 1, 3, 6, 8, 10\})$$

so called because the part *I* corresponds to the intervals of the major scale from the tonic, and the *Fuxian* dichotomy

$$(K/D) = (\{0, 3, 4, 7, 8, 9\} / \{1, 2, 5, 6, 10, 11\})$$

of consonances and dissonances of the Renaissance counterpoint. We will devote this chapter to explain why, in particular the meaning of the counterpoint theorem in this important setup.

3.1 Neuronal Evidence for the Polarity Function

It is entirely natural to select the properties of the (K/D) dichotomy as the first ones to be studied in a systematic and detailed way, being the foundation of a vast corpus of music. In particular, one important milestone is the research conducted by Guerino Mazzola and Heinz-Gregor Wieser regarding the human brain's response to the polarity $p = T^2 \cdot 5$ of the Fuxian dichotomy.

Although this book is not a report on physiological or psychological correlates to music structures, it is important to give an overview of a pronounced evidence of electrophysiological correlates of the consonance-dissonance dichotomy. This is by no means a justification (or even a proof) of the adequacy of the mathematical investigations, but it must be considered as a fundamental relativization of traditional consonance-dissonance theories. These ideas never produce dichotomies but yield

degrees of consonance or dissonance, a quality that is completely irrelevant to the musical counterpoint dichotomy.

The aim of a project at the Neurology Department of the Zürich University Hospital that the epileptologist Heinz-Gregor Wieser, Guerino Mazzola, and their collaborators conducted during the years 1984–88 was to test mathematical principles of classical counterpoint by means of depth EEG responses to musical stimuli. In particular, it was planned to test the relative results of the EEG to different musical inputs (consonances versus dissonances) and not the relationship of responses to musical stimuli versus non-musical stimuli. The latter problem has been investigated with much success by Hellmuth Petsche and his collaborators [73]. It is important to stress that from our results, we do not draw any kind of conclusions concerning a possible genetic nature of musical understanding or a possible universal validity of classical European interval categories. Our investigations show that in some defined regions of the brain of some European humans, certain significant reactions take place—nothing more and nothing less. There is no reason to generalize whatsoever, but there are enough reasons to try to repeat these investigations in other research sites with a comparable infrastructure. This is all the more desirable since the qualitative results of the investigations by Wieser and Mazzola (namely the prominent role of limbic structures for the judgment of musical pleasantness) have been confirmed by others; see [13], for example.

For a more complete report of the results, we refer to [63, 60, 56, 89, 90, 88]. In this short review, we shall restrict ourselves to the two subtests concerning (a) isolated successive intervals and (b) the polarity between simultaneous consonances and dissonances. The results obtained confirm the following hypothesis:

1. There is a significant differentiation of EEG responses to consonant versus dissonant intervals in limbic and auditory brain areas.
2. There is a pronounced sensitivity of these areas in EEG responses to the fundamental polarity between consonances and dissonances.

In particular, using the “spectral participation vector” to quantitatively measure these responses has confirmed our belief that this vector may carry some of the semantic charge of EEG signals.

3.1.1 The EEG Test

The test concerned different contexts of consonances and dissonances as well as the test of the polarity $T^2 \cdot 5$. We used EEG from the scalp (Hess system), stereotactic depth EEG following [87], and multipolar foramen ovale recordings [91]. The tests were applied to patients with the rare case of medically intractable complex partial epilepsy seizure of suspected mediobasal temporal lobe origin and who underwent presurgical evaluation with an eye toward surgical epilepsy therapy. None of the total 13 patients considered the voluntary 30-minute music test through monophonic earphones to be disagreeable.

There are several reasons why, despite the particular state of epileptic patients, the tests remain comparable to tests with the general population, on which the tests cannot be conducted for obvious reasons (depth EEG is a heavy surgical intervention). First, the tests were performed during periods between seizures. Second, localization of the focus gives a good estimation of its possible influence. Third, epileptiform potentials are easily distinguished from others by the expert.

For each patient, we recorded 700 time windows for fast Fourier transform (FFT) spectral analysis with 256 samples per second, each window for different EEG channels, different power windows ($\delta = 0 - 4$ Hz, $\theta = 4 - 8$ Hz, $\alpha = 8 - 14$ Hz, $\beta = 14 - 40$ Hz), and repetitions—totally 11,000 raw spectral data per patient. Unfortunately, the project could not be completed for extrascientific reasons, and only two patients have been thoroughly evaluated. These patients were C.J.-L., a 35-year-old academic, and V.S., a 31-year-old artisan. Both are male Europeans and prefer standard classical, light, and folklore music. Figure 3.1 shows the positions of bipolar depth EEG recordings that we are going to discuss. Notice that C.J.-L.'s recordings RCA, RH, LCA are homologous to V.S.'s recordings 4, 10, 14. Recording RCA lies within the right hippocampus, recording RH is positioned within Heschl's gyrus (auditory cortex), and recording LCA lies within the left hippocampus.

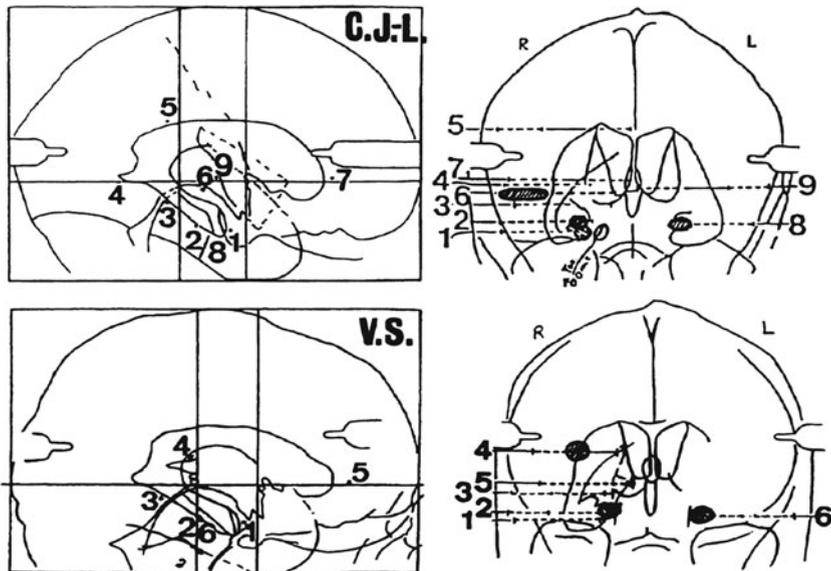


Fig. 3.1 Implantation scheme for the two patients C.J.-L. and V.S. following X-rays showing the topographic position of depth electrodes. The three homologous regions are indicated by dashed ovals. For C.J.-L., RCA is electrode 2/1-3, RH is 6/5-6, and LCA is 8/1-3. For V.S., 4 is electrode 2/1-2, 10 is 4/5-6, and 14 is 6/1-2. (Numbers after the slashes indicate precise positions on the electrodes, where 1 = deepest position and 10 = position near surface).

Nevertheless, we have been able to observe a great deal of visual evidence for EEG response to music stimuli in neo- and archicortical regions of all patients; for details, see [63, 89].

3.1.2 Analysis by Spectral Participation Vectors

We used four sounds for these tests: piano, sine wave, cello (without vibrato), and “test” (a clear, organ-like sound), all synthesized from a Yamaha TX7 synthesizer and CX5M voicing program in order to avoid possible emotional artifacts associated with natural sounds from sociocultural premises. The music program was written on a precursor of the commercial composition software *presto*[®] [54].

The spectral analysis was executed on a CDC Cyber computer. We made use of the *spectral participation vector*

$$S(E) = (P(E), P(E)/P_\delta(E), P(E)/P_\theta(E), P(E)/P_\alpha(E), P(E)/P_\beta(E)),$$

of an event E and its associated *participation value*

$$v(E) = P(E)/P_\theta(E) + P(E)/P_\alpha(E) + P(E)/P_\beta(E),$$

which is a length measure. Here, $P(E)$ is the total spectral power [76], $P_\delta(E)$ is the δ -power of event E , $P_\theta(E)$ is the θ -power, $P_\alpha(E)$ is the α -power, and $P_\beta(E)$ is the β -power, see also [63].

From the results obtained thus far, we conclude that this representation is well-suited to give an adequate picture of a possible semantic charge of EEG signals. It is a measure motivated by, among others, the well-known vigilance-related α -participation $S_\alpha = P(E)/P_\alpha(E)$ observed by Berger. We also use the delta participation $S_\delta = P(E)/P_\delta(E)$, the θ -participation $S_\theta = P(E)/P_\theta(E)$, and the β -participation $S_\beta = P(E)/P_\beta(E)$; the latter has been recognized as being strongly related to higher cognitive brain activity by Giannitrapani (see [32]) and Petsche et al. (who also focus on the γ -band 30 – 50 Hz) [72].

3.1.3 Isolated Successive Intervals

We first focus our attention on a subtest concerning musically isolated successive intervals, i.e., the two tones of an interval are played one after the other without interruption. All the intervals were played in three orders: (1) all consonances, ordered according to their size; all dissonances, also ordered according to their size. (2) All consonances, ordered according to complementarity size (if possible); then all dissonances, ordered according to complementarity (if possible). (3) A mixed succession of all intervals according to a particular dodecahonic all-interval series.

Having fixed a frequency band, θ , say, and an interval with first tone event E_1 and second event E_2 , we consider the quotient

$$Q_\theta(E_1, E_2) = S_\theta(E_1)/S_\theta(E_2)$$

of the theta participations of the first and second tones. If $Q_\theta(E_1, E_2) > 1$ or $Q_\theta(E_1, E_2) < 1$, respectively, then theta participation lowers or rises, respectively, from the first to the second tone. In order to compare these ratios for consonances and dissonances, we take the quotient

$$Q_\theta(K/D) = S_\theta(K)/S_\theta(D)$$

of the mean value $S_\theta(K)$ of all values $Q_\theta(E_1, E_2)$ for consonances (E_1, E_2) and the analogous mean value $S_\theta(D)$ for dissonances. This construction is repeated for all recording positions and the frequency bands θ, α, β ; band δ is omitted since it may be affected by noise. This test was performed four times with patient C.J.-L. and six times with V.S. A one-sided Wilcoxon test shows significantly higher quotients for consonances compared with dissonances, i.e., $Q_\theta(K/D)$ is significantly larger than 1 for many recording positions and frequency bands, see [Figure 3.2](#). Observe the similarity of distribution of these quotients for our patients with respect to homologous recordings and frequency bands. This means that for consonances, participation lowers more when the second tone appears than for dissonances.

3.1.4 Polarity

To test the response to the (K/D) -polarity, we confronted each consonant interval X with all the dissonant intervals Y , and we looked for particular responses in cases where Y was the interval that should correspond to X according to the polarity formula $Y = 2X + 5$. Here, we looked at simultaneous intervals. For each given consonance X , we played a sequence of six confrontations, i.e., immediate successions $(X, X_1), (X, X_2), \dots, (X, X_6)$ of k with each of the six dissonances X_1, X_2, \dots, X_6 , see also [Figure 3.3](#). The duration of each interval was around 0.5 seconds. We looked for the least participation values among the dissonant intervals X_1, X_2, \dots, X_6 , when confronted with a fixed consonance k . We then compared the effective hits to the a priori chance to hit the correct dissonance. This method was applied to every recording position and to the three frequency bands mentioned above. [Figure 3.4](#) shows the numbers compared to 100%, the measure for a priori chance. Due to the small number of samples for this subtest, no statistical tests were applied. However, as [Figure 3.4](#) shows, the results are remarkable and similar for both patients and strongly supports the presence of the (K/D) -polarity as a foundation of contrapuntal processes.

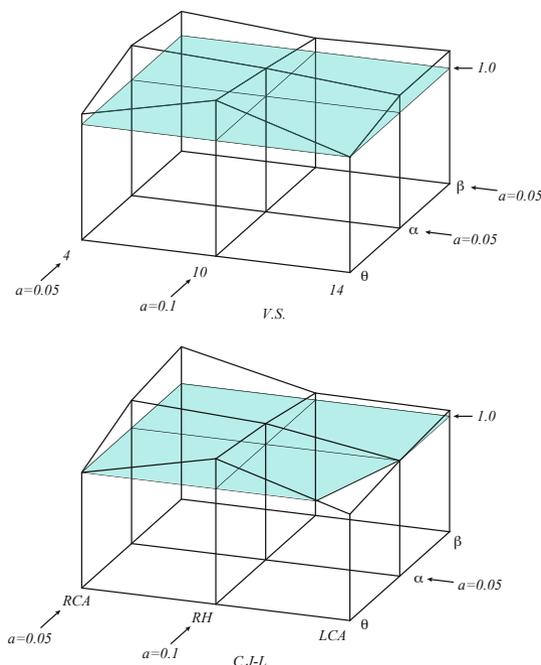


Fig. 3.2 Graphical representation of the quotients $Q_\theta(K/D)$, $Q_\alpha(K/D)$, and $Q_\beta(K/D)$ for all locations and patients C.J.-L. and V.S. These numbers show that for consonances, participation lowers more when the second tone appears, compared with dissonances. The 1-level is indicated in the graphics; observe the places where this level is exceeded.

3.1.5 Music and the Hippocampal Gate Function

It is common knowledge that music and emotions are intimately related. The above results suggest a mechanism that could explain this relationship on the neurophysiological and cognitive level. We have seen that the emotional brain in its hippocampal structures has a pronounced response to elementary structures of harmony and counterpoint—the intervals in their consonance-dissonance dichotomy—and this is so independently of any sound color physics. Now, the classical thesis of Papez and MacLean [52] states that the limbic system, a prominent part of the archicortex, is responsible for emotional human behavior—this is why it is also called the emotional brain. So the hippocampal sensitivity to consonances versus dissonances could relate to the emotional function of music, i.e., of musical intervals, in our case. The question is, how do musical signs, which are by no means emotions by themselves (although Schönberg and other prominent music experts constantly evoke the notes' emotional and erotic life), evoke and signify emotions in humans, and in such a way that the same music may evoke a great variety of such reactions and signifi-

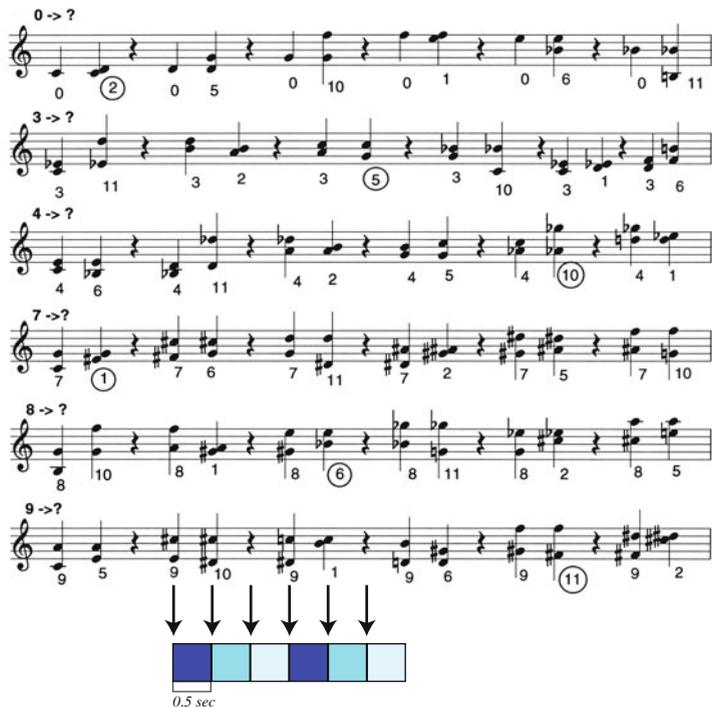


Fig. 3.3 The score of the confrontation test of each consonance with all dissonances; duration of a note in the test is around 0.5 seconds.

cations? Evidently these outputs are the result of a determined sample of music *plus* an individual human ingredient.

The point is that the hippocampal formation has been recognized as a key structure for memory [82]. The neuroscientist Jonathan Winson has proposed a more specific theory of the hippocampal memory function [92], in which he argues that the hippocampus performs a gate function to the subconscious (he even evokes Freud’s “Unbewusstes”), i.e., to memory contents of emotional character. This means that the hippocampus is a structure that plays the role of a gateway to hidden memory contents. It is well known that humans do not have free or controlled access to their memory contents, in particular not on the level of long-term and emotional memory, concerning early childhood, for example.

This suggests that special mechanisms must be activated in order to open the hippocampal gate to unveil locked memory contents. It is plausible from our neurophysiological findings and the gate function of the hippocampus that its musical stimulation could yield such a “key” to open the gate to hidden memory contents. If this were the case, two characteristics of the relationship of music and emotion would be explained at once:

1. The emotional contents are not generated by music but are merely retrieved and evoked from a memory database, which logically explains the individual emotional response to one and the same music.
2. The musical stimulation of the hippocampus is very probably not independent of the human individual who undergoes this process. In other words: If the music is a key, each individual is likely to have his/her own individual key to the “subconscious.” This would explain why there are so many different musical tastes—beyond musical education and culture. This would explain also why it is often a specific tune or musical mood that is the personal preference. If this tune played a role in the encoding of specific emotional memory content, the same key-tune could play a role in the decoding process.

Summarizing, we may conclude that consonant and dissonant intervals and associated harmonic or contrapuntal structures evoke a hippocampus state process that activates a gateway to mainly subconscious memory contents. In other words, Winson’s gate hypothesis of the hippocampal formation must also be stated in the sense of the *existence of a musicogenic key to the gate*.

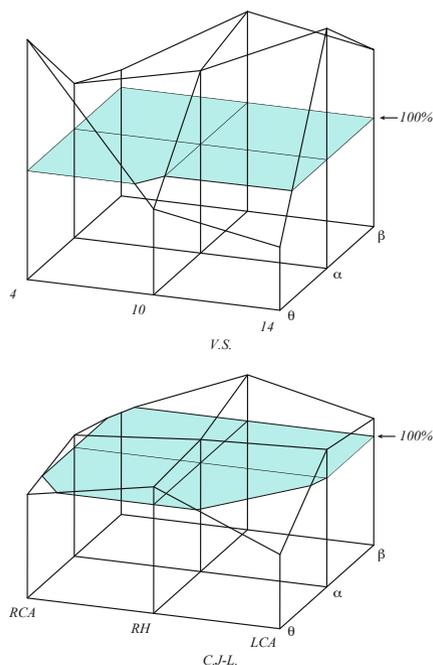


Fig. 3.4 Graphical representation of hitting frequencies for the polarity subtest for patients C.J.-L. and V.S., over three homologous locations and θ, α, β frequency bands. The frequent and pronounced values above 100% show that for both patients there was a strongly affirmative EEG response to the test of the correct values for the polarity. In addition, the topographic/spectral distribution of values for α and β bands is comparable for these patients.

This thesis does not mean that music produces emotions. We state that it only retrieves and reactivates them from a memory database; it acts on the brain like a drug and produces psychological effects. In this metaphor, the ‘chemical formula’ of the musical drug corresponds to the involved musical structure.

3.2 The Counterpoint Theorem Revisited

After the summary of suggestive findings regarding the EEG testing of the polarity function, we can move to more indirect (but perhaps more tangible) confirmations of the adequacy of the mathematical model exposed. The execution of Algorithm 2.1, when applied to the strong dichotomies of \mathbb{Z}_{12} , gives the following result.

Theorem 3.1 (Der große Kontrapunktsatz, Mazzola [53], Hichert [38]). *Let $\Delta = (X/Y)$ be a strong interval dichotomy in \mathbb{Z}_{12} , and let ξ be a consonant interval. Then ξ has at least 42 admissible successors, and it has at least one admissible successor even if we choose the cantus firmus of the successor beforehand (more details appear in Table 3.1). In Table 3.2 appear the general prohibitions apart from repetitions, i.e., the elements $k \neq 0$ of $\varepsilon \cdot \mathbb{Z}_{12}$ such that the succession $x + \varepsilon \cdot k, y + \varepsilon \cdot k$ is always forbidden.*

3.3 The Perspective from the Reduced Strict Style

Various corollaries can be drawn from Theorem 3.1, but perhaps the most striking is related to the classical dichotomy (K/D). It is the only one that results in exactly one general prohibition, which is precisely the use of parallel fifths! This comes as a surprise, since this is deduced from the postulates of the theory, which are not of poetical or historical nature.

Daniel Muzzolini and Guerino Mazzola performed a careful comparison between Fux’s system of rules and the ones derived from this model [61]. Obviously, the traditional set of rules is applicable in ranges greater than the octave, so they reduced it to remain within an octave in order to provide a common arena, obtaining what they called the *reduced strict style*. Once this was done, they discovered that only forbidden parallel fifths and tritone rules have unrestricted validity.

According to their computations, within an ecclesiastical mode there are 287 possible progressions, and 54 of them are forbidden in the reduced strict style, whereas 37 are inadmissible in Mazzola’s model; they coincide in only 21 cases.¹ It is not difficult to prove that, supposing an uninformed agent attempts to guess the prohibitions of the reduced strict style in 37 random trials, on average it would be

¹ This analysis has been recently challenged, specifically with respect to the exact value of these quantities. But the deviation from the numbers discussed here (and the conclusions drawn from them) is minimal. See [6] for details.

| k | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|-------------------------------------------------|---------|------|---------|------|---------|------|---------|------|---------|------|---------|------|
| Forbidden Successors for Dichotomy Class No. 64 | | | | | | | | | | | | |
| 2 | 2 | - | 2 | - | 2 | - | 2 | - | 2 | - | 2 | - |
| 4 | 2,4,7,9 | - | 2,4,7,9 | - | 2,4,7,9 | - | 2,4,7,9 | - | 2,4,7,9 | - | 2,4,7,9 | - |
| 5 | 5,11 | 5,11 | 5,11 | 5,11 | 5,11 | 5,11 | 5,11 | 5,11 | 5,11 | 5,11 | 5,11 | 5,11 |
| 7 | 2,4,7,9 | - | 2,4,7,9 | - | 2,4,7,9 | - | 2,4,7,9 | - | 2,4,7,9 | - | 2,4,7,9 | - |
| 9 | 9 | - | 9 | - | 9 | - | 9 | - | 9 | - | 9 | - |
| 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 |
| Forbidden Successors for Dichotomy Class No. 68 | | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | - | 1 | - | 1 | - | 1 | - | 1 | - | 1 | - |
| 2 | 2,8 | 2,8 | 2,8 | 2,8 | 2,8 | 2,8 | 2,8 | 2,8 | 2,8 | 2,8 | 2,8 | 2,8 |
| 3 | 1,3,5 | - | 1,3,5 | - | 1,3,5 | - | 1,3,5 | - | 1,3,5 | - | 1,3,5 | - |
| 5 | 5 | - | 5 | - | 5 | - | 5 | - | 5 | - | 5 | - |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| Forbidden Successors for Dichotomy Class No. 71 | | | | | | | | | | | | |
| 0 | 0,3 | 3 | 0,3 | 3 | 0,3 | 3 | 0,3 | 3 | 0,3 | 3 | 0,3 | 3 |
| 1 | 1,2,7 | - | 1,2,7 | - | 1,2,7 | - | 1,2,7 | - | 1,2,7 | - | 1,2,7 | - |
| 2 | 2,3 | - | 2,3 | - | 2,3 | - | 2,3 | - | 2,3 | - | 2,3 | - |
| 3 | 2,3 | - | 2,3 | - | 2,3 | - | 2,3 | - | 2,3 | - | 2,3 | - |
| 6 | 3,6 | 3 | 3,6 | 3 | 3,6 | 3 | 3,6 | 3 | 3,6 | 3 | 3,6 | 3 |
| 7 | 1,2,7 | - | 1,2,7 | - | 1,2,7 | - | 1,2,7 | - | 1,2,7 | - | 1,2,7 | - |
| Forbidden Successors for Dichotomy Class No. 75 | | | | | | | | | | | | |
| 0 | 0 | - | - | - | - | 0 | - | - | - | - | - | - |
| 1 | 0,1,4,5 | - | 0,1,4,5 | - | 0,1,4,5 | - | 0,1,4,5 | - | 0,1,4,5 | - | 0,1,4,5 | - |
| 2 | 2,5 | - | 2,5 | - | 2,5 | - | 2,5 | - | 2,5 | - | 2,5 | - |
| 4 | 0,1,4,5 | - | 0,1,4,5 | - | 0,1,4,5 | - | 0,1,4,5 | - | 0,1,4,5 | - | 0,1,4,5 | - |
| 5 | 2,5 | - | 2,5 | - | 2,5 | - | 2,5 | - | 2,5 | - | 2,5 | - |
| 8 | 2,8 | 2 | 2,8 | 2 | 2,8 | 2 | 2,8 | 2 | 2,8 | 2 | 2,8 | 2 |
| Forbidden Successors for Dichotomy Class No. 78 | | | | | | | | | | | | |
| 0 | 0,2,6 | - | 0,2,6 | - | 0,2,6 | - | 0,2,6 | - | 0,2,6 | - | 0,2,6 | - |
| 1 | 1,2 | - | 1,2 | - | 1,2 | - | 1,2 | - | 1,2 | - | 1,2 | - |
| 2 | 1,2 | - | 1,2 | - | 1,2 | - | 1,2 | - | 1,2 | - | 1,2 | - |
| 4 | 1,4 | - | 1,4 | - | 1,4 | - | 1,4 | - | 1,4 | - | 1,4 | - |
| 6 | 0,2,6 | - | 0,2,6 | - | 0,2,6 | - | 0,2,6 | - | 0,2,6 | - | 0,2,6 | - |
| 10 | 1,10 | - | 1,10 | - | 1,10 | - | 1,10 | - | 1,10 | - | 1,10 | - |
| Forbidden Successors for Dichotomy Class No. 82 | | | | | | | | | | | | |
| 0 | 0 | - | - | - | - | 0 | - | - | - | - | - | - |
| 3 | 3,9 | - | 3,9 | - | 3,9 | - | 3,9 | - | 3,9 | - | 3,9 | - |
| 4 | 0,4,8 | - | 0,4,8 | - | 0,4,8 | - | 0,4,8 | - | 0,4,8 | - | 0,4,8 | - |
| 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 8 | 8 | - | 8 | - | 8 | - | 8 | - | 8 | - | 8 | - |
| 9 | 3,9 | - | 3,9 | - | 3,9 | - | 3,9 | - | 3,9 | - | 3,9 | - |

Table 3.1 Forbidden successors of counterpoint intervals for selected representatives of the six strong dichotomies, according to Hichert’s algorithm. For each dichotomy, in the column k appears their respective consonances, and in the columns with heading $j = 0, \dots, 11$ the intervals $j + \varepsilon \cdot \ell$ that are not allowed as successors.

| Dichotomy | Forbidden parallels |
|-----------------------------------------------------------|---------------------|
| $(I/J) = (\{2, 4, 5, 7, 9, 11\} / \{0, 1, 3, 6, 8, 10\})$ | 5, 11 |
| $(\{0, 1, 2, 3, 5, 8\} / \{4, 6, 7, 9, 10, 11\})$ | 0, 2, 8 |
| $(\{0, 1, 2, 3, 6, 7\} / \{4, 5, 8, 9, 10, 11\})$ | None |
| $(\{0, 1, 2, 4, 5, 8\} / \{3, 6, 7, 9, 10, 11\})$ | None |
| $(\{0, 1, 2, 4, 6, 10\} / \{3, 5, 7, 8, 9, 11\})$ | None |
| $(K/D) = (\{0, 3, 4, 7, 8, 9\} / \{1, 2, 5, 6, 10, 11\})$ | 7 |

Table 3.2 General forbidden parallels for selected representatives of the six strong dichotomies.

correct approximately 7 times; its probability of having 21 hits or more is less than 3.2×10^{-7} .

If this probabilistic argument is not deemed serious, we surely can confront the theory with reality and practice. In an analysis of 203 first-species progressions in the *Missa Papae Marcellus* by Giovanni Pierluigi da Palestrina [69], Nieto found that the composer violates in 15 cases the reduced strict style and in only 6 cases Mazzola's model rules. One may think that this is to be expected, since Mazzola's model is more permissive than the reduced strict style, but if the deviations from the mathematical model were to be roughly in the same proportion, we would have 10 or more "mistakes" made by Palestrina with respect to the mathematical model. It is true that Nieto did not provide information on how many times Palestrina failed in both models, but such coincidences argue precisely in favor of the mathematical model.

3.4 The Antipodality of Fuxian and Ionian Dichotomies

Let us examine now the relationship between the (K/D) and (I/J) dichotomies. Recall our visualization from Section 2.1 of \mathbb{Z}_{12} —composed of two types of cycles that go around the scale, namely those of the minor and major thirds—which was devised to highlight the geometric meaning of the action of $\overrightarrow{\text{GL}}(\mathbb{Z}_{12})$. It turns out to be even more useful for geometric purposes, since in fact we constructed a graph of 12 vertices and 24 edges. As in any graph, there is a very natural distance d between two vertices defined as the least number of edges that connect them. This turns the torus of thirds into a metric space.

Problem 3.1. Prove that any $g \in \overrightarrow{\text{GL}}(\mathbb{Z}_{12})$ is an isometry, i.e.,

$$d(g(x), g(y)) = d(x, y).$$

In other words, the distances in the torus of thirds are not altered when we transform it with a symmetry.

Using this distance, we may define the *diameter* of a dichotomy (X/Y) as

$$\delta(X/Y) = \frac{1}{2} \sum_{u, v \in X} d(u, v).$$

and its *span* as

$$\sigma(X/Y) = \sum_{u \in X} d(u, p(u)).$$

Problem 3.2. Prove that the diameter and the span of all the elements of a strong dichotomy class (X/Y) are the same. In other words, if $X' = gX$ and $Y' = gY$, then $\delta(X/Y) = \delta(X'/Y')$ and $\sigma(X/Y) = \sigma(X'/Y')$.

The diameter and span of a dichotomy are measures of how much a dichotomy is “concentrated” or “mixed,” respectively. It turns out that the (K/D) dichotomy is the one with both the smallest diameter and the greatest span, which means their parts are optimally separated and as concentrated as possible. On the other hand, the dichotomy (I/J) is quite the opposite: It has both the greatest diameter and the smallest span; thus their parts are as mixed and dispersed as possible. In this sense, these dichotomies are antipodal.

Some additional facts regarding this antipodality are particularly interesting. First of all, the (K/D) dichotomy has the greatest number of contrapuntal symmetries (17) and the least number of prohibitions (116), whereas this reverses for the (I/J) dichotomy, which has the smallest number of contrapuntal symmetries (10) and the greatest number of interdictions (144).

Furthermore, if we examine the counterpoint theory of all seven-tone scales with the (K/D) dichotomy (which means the cantus firmus and the discantus remain within the selected scale), the major scale provides the greatest freedom of choice for the successors. Conversely, for the (I/J) dichotomy, the major scale is the only one that has contrapuntal dead ends.

Analogously, if we look for seven-tone scales with no cul-de-sacs for the (I/J) dichotomy, no traditional Western scale (namely, major, minor, or melodic minor scale) results. However, the scale $K^* = \{0, 3, 4, 7, 8, 9, 11\}$ does appear, which is the set of Fuxian consonances plus a “leading” note. It is surprising that it is also very close to the raga of seven notes “mayamalavagaula” $\{C, Db, E, F, G, Ab, B\}$ of Indian classical music [21, p. 11]. This is a strong indication of the musical anthropic principle proposed by Mazzola.

3.5 The Fuxian and Riemann Dichotomies

The specific representative (K_7/D_7) of the strong dichotomy class of (K/D) has a very significant algebraic property, previously mentioned in the first chapter.

Proposition 3.1 (T. Noll). *The dichotomy*

$$K_7 = 7.K_1 = \{0, 1, 3, 4, 8, 9\} \subset \mathbb{Z}_{12}$$

is a multiplicative monoid.

Proof. It is a direct consequence of the identities

$$3^2 = 9, 4^2 = 4, 8^2 = 4,$$

$$9^2 = 9, 3 \cdot 4 = 0, 4 \cdot 8 = 8,$$

$$9 \cdot 8 = 0, 3 \cdot 8 = 0, 4 \cdot 9 = 0, 3 \cdot 9 = 3$$

and the fact that $1 \in K_7$.

□

Let us define a special dichotomy following Noll (see [70] and [71]). Let us consider the set of all affine transformations $\overrightarrow{Hom}(\mathbb{Z}_{12}) = \{T^x \cdot y : x, y \in \mathbb{Z}_{12}\}$ of \mathbb{Z}_{12} , and the subset $R \subset \overrightarrow{Hom}(\mathbb{Z}_{12})$ of symmetries that map the major triad $Y = 7\{0, 4, 7\} = \{0, 4, 1\}$ (where its intervals are seen as multiples of fifths) on itself. A brief calculation with the 144 available symmetries confirms that

$$R = \{T^0 \cdot 0, T^1 \cdot 0, T^4 \cdot 0, T^{11} \cdot 1, T^{10} \cdot 3, T^8 \cdot 4, T^8 \cdot 8, T^7 \cdot 9\}.$$

Note that the linear parts of the elements of R are precisely the elements of K_7 . The monoid generated by R

$$\mathcal{R} = \langle R \rangle,$$

is a marked dichotomy in the set of all symmetries of \mathbb{Z}_{12} , as we now show.

Lemma 3.1. *We have*

$$|\mathcal{R}| = 72.$$

Proof. Since $T^{11} \cdot 1 = T^{-1} \cdot 1 \in R$, we have

$$T^k \cdot 1 = (T^{-1} \cdot 1)^{12-k} \in R$$

for all $k \in \mathbb{Z}_{12}$. We also have $T^0 \cdot v \in \mathcal{R}$ for each $v \in K_7$, because for any $T^u \cdot v \in R$ it is true that

$$T^0 \cdot v = T^{-u} \cdot 1 \circ T^u \cdot v \in \mathcal{R},$$

and we have already observed that every $v \in K_7$ appears as the linear part of an element of R . Given $T^{u_1} \cdot v_1, T^{u_2} \cdot v_2 \in R$,

$$T^{u_1} \cdot v_1 \circ T^{u_2} \cdot v_2 = T^{u_1+v_1u_2} \cdot v_1v_2 \in \mathcal{R}$$

and $v_1v_2 \in K_7$, because of Proposition 3.1. Hence

$$\mathcal{R} = T^{\mathbb{Z}_{12}} \circ K_7$$

and therefore $12 \cdot 6 = 72$, since 12 different translations can be composed with 6 different linear parts. \square

The monoid \mathcal{R} is the so-called *Riemann dichotomy*. The following result relates a subset of the endomorphisms of $K_7[\varepsilon]$ (the consonant intervals from the fifth perspective) with the Riemann dichotomy.

Proposition 3.2 (T. Noll). *The set W of the endomorphisms of $K_7[\varepsilon]$ of the form $T^x \cdot y$ coincide with \mathcal{R} .*

Proof. Take any $T^x(k) \in \mathcal{R}$ (where $k \in K_7$) and $a + \varepsilon.b \in K_7[\varepsilon]$. We have

$$T^x \cdot k(a + \varepsilon.b) = (ka + x) + \varepsilon.kb \in K_7[\varepsilon]$$

because $kb \in K_7$, K_7 being a multiplicative monoid, whence $\mathcal{R} \subseteq W$.

Conversely, if $T^x(y) \in W$, then for any $a + \varepsilon.1 \in K_7[\varepsilon]$ the following holds:

$$T^x \cdot y(a + \varepsilon) = (a + x) + \varepsilon.y \in K_7[\varepsilon],$$

which implies that $y \in K_7$ and henceforth $T^x(y) \in \mathcal{R}$. □

Let us consider the bijection

$$\begin{aligned} \nu : \overrightarrow{Hom}(\mathbb{Z}_{12}) &\longrightarrow \mathbb{Z}_{12}[\varepsilon], \\ T^a \cdot b &\longmapsto a + \varepsilon.b. \end{aligned}$$

We see that

$$\nu^{-1}(K_7[\varepsilon]) = T^{\mathbb{Z}_{12}} K_7 = \mathcal{R} = W.$$

Corollary 3.1. *Every consonant interval is isomorphic to a product of endomorphisms of the major triad, and isomorphic to an endomorphism of the set of consonant contrapuntal intervals $K_7[\varepsilon]$.*

Corollary 3.2. *Let $f = 0 + \varepsilon.1$ be the interval of a fifth. For any consonant interval $c = x + \varepsilon.k \in K_7[\varepsilon]$ there exists a sequence*

$$t_i = T^{x_i} \cdot y_i, \quad i = 1, \dots, m$$

of automorphisms $t_i : Y \rightarrow Y$ such that

$$c = t_m \circ t_{m-1} \circ \dots \circ t_1(f).$$

Chapter 4

Graphs

A counterpoint composition is like traveling through a city: We move from place to place (the intervals) driving along roads (the steps), as shown on [Figure 4.1](#). Some routes may allow traffic in both directions, others may be one-way roads, and a few may be completely closed to vehicles. The network is designed to optimize traffic in the same way as counterpoint rules regulate musical circulation across intervals. The duty and freedom of the composer is to choose the appropriate route in this system.

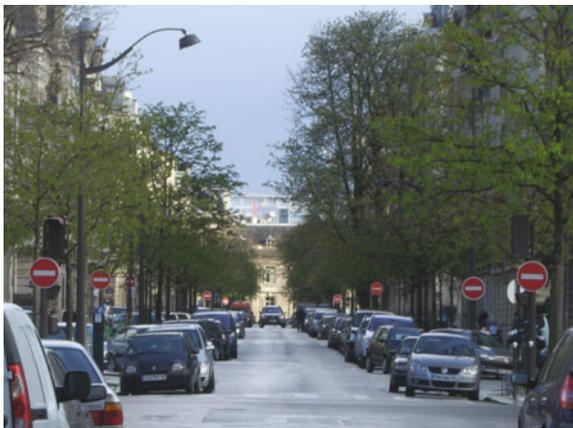


Fig. 4.1 Counterpoint rules work like traffic signs. A musical composition is analogous to a ride through the allowed routes of a city. The composer is the driver, the listener the passenger.

First, a remark about the notation we will use throughout this chapter and the two next ones. In all examples and figures, contrapuntal consonances $\xi = x + \varepsilon.k$ will be written k_x for short. Sets of consonances with related cantus firmus will be designated by

$$(k_1, \dots, k_m)_{x_0|\Delta x} := \{x + \varepsilon.k_i \in X[\varepsilon] : \exists j \in \mathbb{Z}_n : x = x_0 + j \cdot \Delta x \text{ and } i = 0, \dots, m\}. \quad (4.1)$$

If not all consonances share the same cantus firmus pattern, the set is split into homogeneous subsets. As explained in Figure 4.2, the notation becomes a concatenation of k disjunct subsets,

$$(k_{i_1}, \dots, k_{i_{m_1}})_{x_1 | \Delta x_1} \cdots (k_{i_k}, \dots, k_{i_{m_k}})_{x_k | \Delta x_k}. \tag{4.2}$$

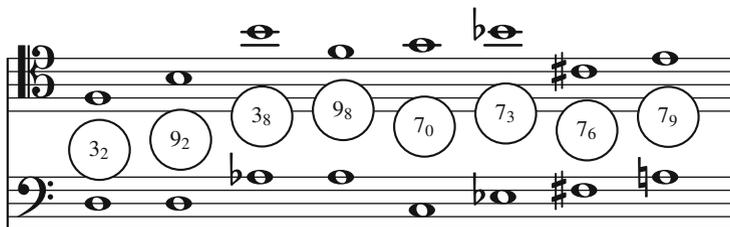


Fig. 4.2 Encoding of eight contrapuntal consonances of the Fuxian world. The discantus lies on the upper staff, the cantus firmus on the lower staff. With C as the reference pitch class, the minor third anchored to D2 is written 3_2 , the major sixth 9_2 . The pair of intervals anchored to Ab2 become 3_8 and 9_8 . The last four fifths are anchored to different cantus firmi and need to be set apart. The entire set is written $(3, 9)_{2|6}(7)_{0|3}$.

4.1 Counterpoint Worlds

Which components define a counterpoint system? Basically, the choice of a strong dichotomy specifies the $2k$ available notes, the k allowed intervals, a symmetry, and by means of the counterpoint theorem described in Section 3.2, the allowed moves from one consonance to the next. A counterpoint world thus consists of $(2k)^2$ contrapuntal intervals along with two classification schemes: one for intervals, split into consonances and dissonances, and one for the $(2k)^4$ ordered pairs of intervals, split into allowed and forbidden steps.

A counterpoint system as a whole may be too big a structure to be manipulated and transformed into another one. However, when transforming musical compositions in practice, one seldom uses the entire set of available consonances. Restricting the problem to only those intervals that appear in a given counterpoint can help in finding a counterpoint transformation, despite the absence of “global” solutions mentioned in Section 5.4.2.

Definition 4.1. Let $\Delta = (X/Y)$ be a strong dichotomy of \mathbb{Z}_{2k} with polarity function p_Δ , where $2k \geq 6$ is even, and let S be any subset of $\mathbb{Z}_{2k}[\varepsilon]$. The global polarity function $p_\Delta^\bullet : \mathbb{Z}_{2k}[\varepsilon] \rightarrow \mathbb{Z}_{2k}[\varepsilon]$ is defined as a mapping of the contrapuntal intervals

to their dual intervals,

$$p_{\mathcal{A}}^{\bullet}(x + \varepsilon.i) = x + \varepsilon.p_{\mathcal{A}}(i), \quad (4.3)$$

and the set of relevant contrapuntal intervals

$$\bar{S} := S \cup p_{\mathcal{A}}^{\bullet}(S) \quad (4.4)$$

as a symmetric extension to S .

A counterpoint world \mathcal{W} is a triple $(\kappa, \sigma, p_{\mathcal{A}}^{\bullet})$ formed by the polarity function and two indicator functions.

1. The interval indicator function $\kappa : \bar{S} \rightarrow \{0, 1\}$ picks the consonances:

$$\kappa(\zeta) := \begin{cases} 0 & \text{if } \zeta \in Y[\varepsilon] \cap \bar{S}, \\ 1 & \text{if } \zeta \in X[\varepsilon] \cap \bar{S}. \end{cases} \quad (4.5)$$

2. The step indicator function $\sigma : \bar{S} \times \bar{S} \rightarrow \{0, 1\}$ picks the allowed successions:

$$\sigma(\zeta_0, \zeta_1) := \begin{cases} 0 & \text{if } \zeta_1 \notin X_{\zeta_0} \cap \bar{S} \text{ or } \zeta_0 \in Y[\varepsilon] \cap \bar{S}, \\ 1 & \text{if } \zeta_1 \in X_{\zeta_0} \cap \bar{S}, \end{cases} \quad (4.6)$$

where X_{ζ_0} is the set of allowed successors to the consonance ζ_0 , as output by Hichert's algorithm of Section 2.4.2.

A counterpoint world is said to be global if \bar{S} is equal to the entire set of contrapuntal intervals, and local if \bar{S} is a proper subset.

In case of a global world, the polarity function is redundant: It is determined by the list of consonances. Nevertheless, a proper subset of consonances may not suffice to deduce a unique polarity function, hence its inclusion in the definition above, which serves for both global and local worlds.

Defining a transformation ψ between two (possibly the same) counterpoint worlds $\mathcal{A} = (X/Y)$ and $\mathcal{A}' = (X'/Y')$ means finding a mapping of contrapuntal intervals, the very building blocks of counterpoint, that faithfully preserves the contrapuntal structure, i.e., the composition rules:

1. Consonances are mapped to consonances,

$$\forall \xi \in \bar{S}, \quad \xi \in X[\varepsilon] \Rightarrow \psi(\xi) \in X'[\varepsilon]. \quad (4.7)$$

2. Dissonances are mapped to dissonances,

$$\forall \xi \in \bar{S}, \quad \xi \in Y[\varepsilon] \Rightarrow \psi(\xi) \in Y'[\varepsilon]. \quad (4.8)$$

3. Allowed steps are mapped to allowed steps,

$$\forall \xi_0, \xi_1 \in \bar{S}, \quad \xi_1 \in X_{\xi_0} \Rightarrow \psi(\xi_1) \in X'_{\psi(\xi_0)}. \quad (4.9)$$

4. Forbidden steps are mapped to forbidden steps,

$$\forall \xi_0, \xi_1 \in \bar{S}, \quad \xi_1 \notin X'_{\xi_0} \Rightarrow \psi(\xi_1) \notin X'_{\psi(\xi_0)}. \quad (4.10)$$

Definition 4.2. Let $\mathcal{W} = (\kappa, \sigma, p_{\Delta}^{\bullet})$ and $\mathcal{W}' = (\kappa', \sigma', p_{\Delta'}^{\bullet})$ be two counterpoint worlds, and $\bar{S} = \text{dom}(\kappa)$, $\bar{S}' = \text{dom}(\kappa')$ their respective relevant contrapuntal intervals. A counterpoint world morphism is a set function $\psi : \bar{S} \rightarrow \bar{S}'$ compatible with the contrapuntal structures:

1. It commutes with the interval indicators.

$$\begin{array}{ccc} \bar{S} & \xrightarrow{\psi} & \bar{S}' \\ \searrow \kappa & & \swarrow \kappa' \\ & \{0, 1\} & \end{array} \quad (4.11)$$

2. It commutes with the step indicators.

$$\begin{array}{ccc} \bar{S} \times \bar{S} & \xrightarrow{\psi \times \psi} & \bar{S}' \times \bar{S}' \\ \searrow \sigma & & \swarrow \sigma' \\ & \{0, 1\} & \end{array} \quad (4.12)$$

3. It commutes with the polarities, swapping consonances and dissonances.

$$\begin{array}{ccc} \bar{S} \cap X & \xrightarrow{\psi} & \bar{S}' \cap X' \\ p_{\Delta}^{\bullet} \uparrow & & \uparrow p_{\Delta'}^{\bullet} \\ \bar{S} \cap Y & \xrightarrow{\psi} & \bar{S}' \cap Y' \end{array} \quad (4.13)$$

Counterpoint worlds together with counterpoint world morphisms define the category $\mathbb{C}\mathbb{W}\text{orld}$ of counterpoint worlds. The identity set function on any set of relevant consonances defines an identity world morphism on any world. It follows from the diagrams (4.11), (4.12), and (4.13) that commutativity is preserved under morphism composition, and the underlying set functions ensure associativity of the composition law.

A counterpoint world only provides the framework for composition; it is not the result of the act of composing, as the double meaning of the word *counterpoint* may suggest. The next definition defines formally a musical composition in this context.

Definition 4.3. Let $2k \geq 6$ be the number of tones. A counterpoint $p := (\zeta_0, \dots, \zeta_n)$ is a finite ordered sequence of contrapuntal intervals where $\zeta_s \in \mathbb{Z}_{2k}[\varepsilon]$, $0 \leq s \leq n$, and $n \in \mathbb{N}$ is the number of steps.

Given a particular counterpoint world $\mathcal{W} = (\kappa, \sigma, p_{\Delta}^{\bullet})$, the two indicator functions can tell if a counterpoint p is a valid composition. The following conditions must be met:

$$\begin{aligned} \kappa(\zeta_s) &= 1, & \forall s \in \{0, \dots, n\}, \\ \sigma(\zeta_s, \zeta_{s+1}) &= 1, & \forall s \in \{0, \dots, n-1\}. \end{aligned} \tag{4.14}$$

A counterpoint morphed under a world morphism ψ results from the individual mapping of every interval of the original counterpoint p :

$$p' := (\psi(\zeta_0), \dots, \psi(\zeta_n)). \tag{4.15}$$

Its validity in the target world \mathcal{W}' can be verified using the target indicator functions κ' and σ' . If p is a valid composition in \mathcal{W} , compatibility of ψ with the indicator functions in diagrams (4.11) and (4.12) would be enough to produce a perfectly valid counterpoint p' in \mathcal{W}' . There are at least three reasons for requiring the additional symmetry in (4.13).

1. On a conceptual level, the polarity function is fully part of the contrapuntal structure. It expresses the duality of consonances and dissonances. If we had chosen the dissonances instead of the consonances to be the allowed intervals, we could have derived a symmetrical set of allowed steps linking the dissonances.¹ Choosing between consonances and dissonances is arbitrary and does not have any consequences for the structure derived.
2. On a practical level, linking consonances and dissonances allows one to keep track of deviations from rules. Imagine a counterpoint integrating dissonances into its sequence. The polarity functions provide a rule for mapping dissonances consistently with the consonances. We can therefore define ways of transforming counterpoints from one world to another, handling not only valid compositions, but also any “forbidden” intervals or steps in a consistent, systematic fashion.
3. On a technical level, requiring compatibility with the polarity functions greatly simplifies calculations, as explained in more detail at the end of Section 4.2.

4.2 Strict Digraphs

In the mathematical model introduced in Section 2.4, rules of composition always involve two successive consonances,² so that the step indicator function of a counterpoint world always defines a binary relation on the finite product set $\bar{S} \times \bar{S}$. It can therefore be interpreted as an adjacency relations in a digraph, or directed graph (see mathematical appendix, Section A.2).

¹ The software described in Chapter 6 enables experimentation with this kind of anti-Fuxian counterpoint. Simply choose the strong dichotomy containing only Fuxian dissonances: $\{1, 2, 5, 6, 10, 11\}$. Instead of an interdiction for parallel fifths, it will yield an interdiction for parallel minor seconds, which are dual to fifths ($p_d(5) = 2 + 5 \cdot 7 = 1$ modulo 12).

² Cases exist where rules of traditional first-species counterpoint deal with a succession of three intervals, but most of the corpus handles pairs of successive intervals.

Definition 4.4. Let $\mathcal{W} = (\kappa, \sigma, p_d^*)$ be a counterpoint world. Its associated strict digraph $D = (V_D, A_D)$ is a directed graph whose vertex set is the consonant part of the relevant contrapuntal intervals

$$V_D := \kappa^{-1}(1), \tag{4.16}$$

and the arrow set the forbidden steps

$$A_D := \sigma^{-1}(0) \cap (\kappa^{-1}(1) \times \kappa^{-1}(1)). \tag{4.17}$$

A consonance can never follow itself. This interdiction imposes a loop on each vertex, turning strict digraphs into *pseudographs*. However, all arrows joining different vertices are simple; there are no multiple, or parallel, arrows.

Since a valid counterpoint should avoid any forbidden steps, in the context of a strict digraph $D = (V_D, A_D)$, a musical composition p translates to a directed walk (multiple uses of steps or intervals are allowed) through the complement of the arrow set A_D^c . The definition above reflects the composer’s focus. One is supposed to limit the use of intervals to consonances and avoid forbidden steps, which are much less numerous than the allowed steps. Paying attention to exceptions rather than to the norm yields a better understanding and representation of contrapuntal structures. There lies the information, and traffic signs work the same way (see Figure 4.3).



Fig. 4.3 Avoid overwhelming redundancy in visual information. How could a driver be aware of dangers or prohibitions if traffic signs, instead of offering warnings, showed every possible route?

Definition 4.5. Let $D = (V_D, A_D)$ and $D' = (V_{D'}, A_{D'})$ be two strict digraphs. A strict digraph morphism, or *strict morphism* for short, is a set function $\phi : V_D \rightarrow V_{D'}$ that is structure preserving, i.e., it preserves both adjacency and non-adjacency:

$$\forall \xi_0, \xi_1 \in V_D, \quad (\xi_0, \xi_1) \in A_D \Leftrightarrow (\phi(\xi_0), \phi(\xi_1)) \in A_{D'}. \quad (4.18)$$

Strict digraphs and strict digraph morphisms are not merely a graph-theoretical reformulation of counterpoint worlds and counterpoint morphisms. Counterpoint worlds are the tools composers can deal with. Their definition appears as a straightforward formalization of contrapuntal structures. On the other hand, strict digraphs are tools for a computer-based implementation and the first step toward size reduction of the counterpoint transformation problem described in Section 5.2. Ignoring the dissonances divides the digraph's order by two; using arrows to denote the less numerous forbidden steps turns it into a sparse digraph. The less intuitive character of the definition is balanced by the practical advantage of fewer data for algorithms to process.

However, it is essential to ensure that the reduction of a counterpoint world to its computer-friendly graph-theoretical representation is lossless, i.e., that it is possible to reconstruct a counterpoint world morphism from a morphism.

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\psi} & \mathcal{W}' \\ \downarrow & & \downarrow \\ D & \xrightarrow{\phi} & D' \end{array} \quad (4.19)$$

The next lemma gives a concrete procedure for constructing a counterpoint world morphism, given a strict morphism.

Lemma 4.1. *Let $\mathcal{W} = (\kappa, \sigma, p_A^\bullet)$ and $\mathcal{W}' = (\kappa', \sigma', p_{A'}^\bullet)$ be two counterpoint worlds, $\bar{S} = \text{dom}(\kappa)$ and $\bar{S}' = \text{dom}(\kappa')$ their respective sets of relevant contrapuntal intervals, and D, D' their respective associated strict digraphs. Let $\phi : V_D \rightarrow V_{D'}$ be a strict morphism. Then any function $\psi : \bar{S} \rightarrow \bar{S}'$ defined by the rule*

$$\psi(\zeta) := \begin{cases} \phi(\zeta) & \text{if } \zeta \in \kappa^{-1}(1), \\ p_{A'}^\bullet \circ \phi \circ p_A^\bullet(\zeta) & \text{if } \zeta \in \kappa^{-1}(0), \end{cases} \quad (4.20)$$

is a counterpoint world morphism.

Proof. We must verify the conditions that ψ must satisfy to be a counterpoint world morphism. Because the definition (4.20) handles the two interval types separately, we will treat consonances and dissonances separately.

1. Consonances are sent to consonances. By definition of a strict digraph (4.16), the vertex set contains only consonances: $\forall \zeta \in \bar{S}, \zeta \in V_D \Rightarrow \kappa(\zeta) = 1$. The image of ϕ lies in the target vertex set, i.e., $\forall \zeta \in \bar{S}, \zeta \in V_D \Rightarrow \phi(\zeta) \in V_{D'}$. Because the target vertex set contains only consonances,

$$\forall \zeta \in \bar{S}, \quad \kappa(\zeta) = 1 \Rightarrow \kappa' \circ \phi(\zeta) = 1.$$

2. Dissonances are sent to dissonances. A dissonance lies outside the domain of ϕ : $\forall \zeta \in \bar{S}, \zeta \in V_D^c \Rightarrow \kappa(\zeta) = 0$. Because the strict morphism ϕ only maps con-

sonances, a dissonance ζ must first be turned into a consonance by means of the global polarity function, which swaps dissonances and consonances: $\forall \zeta \in \bar{S}, \zeta \in V_D^c \Rightarrow p_A^\bullet(\zeta) \in V_D$. The dual consonance mapped under ϕ is again a consonance, which needs to be sent again to the target dissonances,

$$\forall \zeta \in \bar{S}, \quad \kappa(\zeta) = 0 \Rightarrow \kappa' \circ p_{A'}^\bullet \circ \phi \circ p_A^\bullet(\zeta) = 0.$$

Therefore, the function ψ does not mix the two types of intervals.

3. Allowed steps are sent to allowed steps. An allowed step involves two consonances, and according to the definition of a strict digraph,

$$\forall \zeta_0, \zeta_1 \in \bar{S}, \quad \sigma(\zeta_0, \zeta_1) = 1 \Rightarrow \begin{cases} \zeta_0, \zeta_1 \in V_D, \\ (\zeta_0, \zeta_1) \notin A_D. \end{cases}$$

Because the strict morphism ϕ always preserves non-adjacency,

$$\forall \zeta_0, \zeta_1 \in \bar{S}, \quad \sigma(\zeta_0, \zeta_1) = 1 \Rightarrow \sigma'(\phi(\zeta_0), \phi(\zeta_1)) = 1.$$

4. Forbidden steps are sent to forbidden steps. Any forbidden step between two consonances ξ_0 and ξ_1 generates an arrow $(\xi_0, \xi_1) \in A_D$. By definition, ϕ preserves adjacency,

$$\forall \zeta_0, \zeta_1 \in V_D, \quad \sigma(\zeta_0, \zeta_1) = 0 \Rightarrow \sigma'(\phi(\zeta_0), \phi(\zeta_1)) = 0.$$

If at least one of the contrapuntal intervals composing the step is a dissonance, the step is forbidden, as defined in (4.6): $\forall \zeta_0, \zeta_1 \in \bar{S}, \zeta_0 \in V_D^c$ or $\zeta_1 \in V_D^c \Rightarrow \sigma(\zeta_0, \zeta_1) = 0$. A dissonance being sent to a dissonance, the target step will be forbidden too:

$$\forall \zeta_0, \zeta_1 \in \bar{S}, \quad \sigma(\zeta_0, \zeta_1) = 0 \Rightarrow \sigma'(\phi(\zeta_0), \phi(\zeta_1)) = 0.$$

5. Commutativity with polarities. We use the fact that polarity functions are their own inverses. First, for the consonances,

$$\begin{aligned} \forall \xi \in V_D, \quad \psi \circ p_A^\bullet(\xi) &= (p_{A'}^\bullet \circ \phi \circ p_A^\bullet) \circ p_A^\bullet(\xi) \\ &= (p_{A'}^\bullet \circ \phi) \circ (p_A^\bullet \circ p_A^\bullet)(\xi) \\ &= p_{A'}^\bullet \circ \phi(\xi) \\ &= p_{A'}^\bullet \circ \psi(\xi). \end{aligned}$$

The case of the dissonances is treated in a similar way:

$$\begin{aligned} \forall \eta \in V_D^c, \quad \psi \circ p_A^\bullet(\eta) &= \phi \circ p_{A'}^\bullet(\eta) \\ &= (p_{A'}^\bullet \circ p_{A'}^\bullet) \circ (\phi \circ p_A^\bullet)(\eta) \\ &= p_{A'}^\bullet \circ (p_{A'}^\bullet \circ \phi \circ p_A^\bullet)(\eta) \\ &= p_{A'}^\bullet \circ \psi(\eta). \end{aligned}$$

Since consonances and dissonances partition \bar{S} , all cases are covered. The set function ψ faithfully maps the contrapuntal structure and is hence a counterpoint world morphism. \square

Being able to use a strict morphism instead of a counterpoint world morphism without loss of information comes with a price. Having formulated more restrictive definitions, we will miss some possible transformations of counterpoints. If we had only the production of valid counterpoint compositions in mind, we could have considered the allowed steps as the defining elements of arrows and simply used digraph homomorphisms. They preserve adjacency, which ensures that no allowed step in the source counterpoint becomes a forbidden step in the target counterpoint (but originally invalid steps may be mapped to an allowed one). Our setting is less tolerant but is motivated by mathematical (see Section 4.3.3) and technical (see Section 5.2.3) reasons. The requirements added to the definitions that go beyond the sole preservation of counterpoint rules are summarized below.

1. Commutation of the counterpoint world morphism with the polarity functions in (4.13). It is safe to work only on the consonant part and, where needed, to use the polarity functions to reconstruct the dissonant part. If a real-world counterpoint happens to use a dissonance η , which lies outside the vertex set of the strict digraph, the polarities allow us to still use the strict digraph homomorphism by taking care of its dual consonance ξ instead.

$$\begin{array}{ccc}
 \eta & & \eta' \\
 \downarrow p_{\Delta}^{\bullet} & & \uparrow p_{\Delta'}^{\bullet} \\
 \xi & \xrightarrow{\phi} & \xi'
 \end{array} \tag{4.21}$$

2. Preservation of non-adjacency. Digraph homomorphisms are usually defined as preserving arrows³ and are free to map non-adjacent vertices to adjacent ones. Because we have chosen to include only forbidden steps in the strict arrow sets defined in (4.17), such a standard homomorphism could map an allowed step to a forbidden one, transforming valid counterpoints into invalid ones. It is thus necessary to be more restrictive and preserve not only the arrow set, but its complement as well. Looking at condition (4.18), we can see that the usual homomorphism definition is given by the simple implication, i.e. an arrow in the source digraph has to be mapped to an arrow in the target digraph, but we also require its converse to be true when defining strict digraph morphisms.

Strict digraphs and strict morphisms form a subcategory of the category of directed graphs. Note that we have defined a functor from the category of strict digraphs, denoted \mathfrak{D} istrict, to that of counterpoint worlds:

³ See the definition in Section A.2 or standard texts such as [33] on page 444.

$$\begin{aligned}
D &\mapsto \mathcal{W}, \\
D' &\mapsto \mathcal{W}', \\
\phi &\mapsto \psi.
\end{aligned}
\tag{4.22}$$

4.3 Quotient Digraphs

Many consonances show similar patterns regarding their allowed successors. A classical example is the fifth in the Fuxian system: Any consonance can follow a fifth, excepted another fifth. These similarities suggest the grouping of vertices of a strict digraph according to connectivity and neighborhood. Aggregated consonances can become the vertices of new digraphs, which show lower order and size. The next section lists several ways of simplifying the contrapuntal structure.

4.3.1 Vertex Partitions

Let \sim be an equivalence relation defined on a set S . The associated canonical map $\pi_{\sim} : S \rightarrow 2^S$ projects the set S to its partition S/\sim : $\pi_{\sim}(x) = [x]_{\sim}$.

Definition 4.6. Let $D = (V_D, A_D)$ be a strict digraph and \sim_q an equivalence relation defined on the vertex set V_D . The quotient digraph $Q = (V_Q, A_Q)$ of the strict digraph D associated with the equivalence relation \sim_q is the directed graph defined by the quotients of the vertex and arrow sets of D :

$$V_Q := \pi_{\sim_q}(V_D), \tag{4.23}$$

$$A_Q := \pi_{\sim_q} \times \pi_{\sim_q}(A_D). \tag{4.24}$$

Quotient vertices are called components of the original digraph D .

The quotient vertex set is simply the partition generated by the equivalence relation. A quotient arrow exists as soon as an arrow connects any two members of the components:

$$A_Q = \{(q_0, q_1) \in V_Q^2 : \exists(\xi_0, \xi_1) \in A_D \text{ such that } \xi_0 \in q_0 \text{ and } \xi_1 \in q_0\}. \tag{4.25}$$

The projection function π_{\sim} is thus a digraph homomorphism in the usual sense. Quotient digraphs are of the same type as strict digraphs: They show systematic loops and simple arrows.

Definition 4.7. Let $D = (V_D, A_D)$ and $D' = (V_{D'}, A_{D'})$ be two strict digraphs and $\phi : V_D \rightarrow V_{D'}$ a strict digraph morphism. Let \sim_q and $\sim_{q'}$ be two equivalence relations defined on V_D and $V_{D'}$ respectively. They are said to be compatible if there exists a digraph homomorphism $\phi_Q : V_Q \rightarrow V'_{Q'}$, called a quotient morphism, for which the diagram

$$\begin{array}{ccc}
 V_D & \xrightarrow{\phi} & V_{D'} \\
 \pi_{\sim q} \downarrow & & \downarrow \pi_{\sim q'} \\
 V_Q & \xrightarrow{\phi_Q} & V'_Q
 \end{array} \tag{4.26}$$

commutes.

Not every pair of equivalence relations is compatible, i.e., yields a well-defined quotient morphism. Compatibility in this context means that all vertices of a class in the source digraph have to be mapped to vertices of the same class in the target digraph:

$$\phi_Q(q) := \pi_{\sim'} \circ \phi \circ \pi_{\sim}^{-1}(q). \tag{4.27}$$

Choosing the same⁴ equivalence relations on both digraphs will always ensure (4.26) holds. Once a quotient morphism is well-defined, it is also unique.

4.3.2 Component Connections

A strict digraph's arrows carry all the information on rules governing the succession of consonances. They form a sophisticated network of interdictions, harder to remember and understand than a short list of consonances. One way of simplifying the problem is to use quotient digraphs, which provide a summary of the contrapuntal structures.

Common definitions of connectivity in digraphs include *weak* and *strong* components, which will be discussed in Sections 4.3.2.2 and 4.3.2.3, respectively.⁵ These are not the only partitions possible—we will also consider three other possibilities.

4.3.2.1 Full Partitions

The broadest possible partition is the trivial partition built on a single component encompassing the entire vertex set.

Definition 4.8. Let $D = (V_D, A_D)$ be a strict digraph. The full equivalence relation \sim_f on the vertex set V_D is defined as the universal equivalence.

$$\forall \xi_0, \xi_1 \in V_D, \quad \xi_1 \sim_f \xi_0 \tag{4.28}$$

Such a partition is used to designate the digraph D itself and to function as a root when we organize components into a tree, as will be done in Section 5.3.3.

For every counterpoint world, the quotient graph $F = (\{f\}, \{(f, f)\})$ is a complete digraph K_1 , see [Figure 4.4](#).

⁴ Two relations based on the same definition, but applied to two different vertex sets.

⁵ See also section 1.5 on page 16 in [11] or section 12.1 on page 494 in [33].

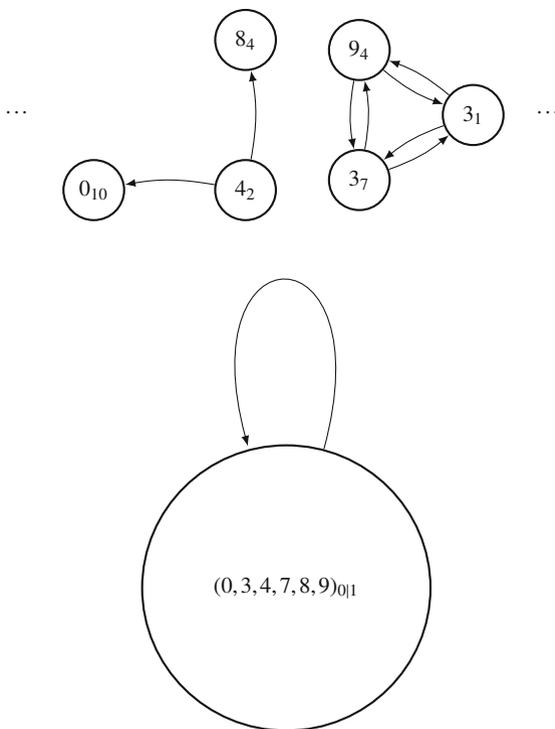


Fig. 4.4 The full quotient digraph as a complete singleton K_1 (below). Presence, absence, and orientation of arrows in the strict digraph (above) do not play a role. All Fuxian consonances are identified whatever their connections. For the sake of legibility, loops of the strict digraph are not represented.

A full quotient morphism will be a mapping from a single component to another. Its basic purpose is to indicate whether a corresponding strict morphism exists between the two worlds.

4.3.2.2 Weak Partitions

Definition 4.9. Let $D = (V_D, A_D)$ be a strict digraph. The weak equivalence relation on the vertex set V_D identifies two vertices as soon as they can be connected by a chain of arrows, whatever their orientation. For any $i \in \{0, \dots, n - 1\}$,

$$\forall \xi_0, \xi_n \in V_D, \quad \xi_n \sim_w \xi_0 \Leftrightarrow \exists \xi_1, \dots, \xi_{n-1} \in V_D : (\xi_i, \xi_{i+1}) \in A_D \text{ or } (\xi_{i+1}, \xi_i) \in A_D \quad (4.29)$$

The weak partition corresponds to the connected components of the underlying undirected graph. Equation (4.29) yields an equivalence relation because it applies to strict digraphs: The loops are necessary for reflexivity, $\forall \xi \in V_D, (\xi, \xi) \in A_D$. The

weak quotient directed graph W , called weak graph for short, is a union of complete graphs K_1 , one per component, as shown in Figure 4.5.

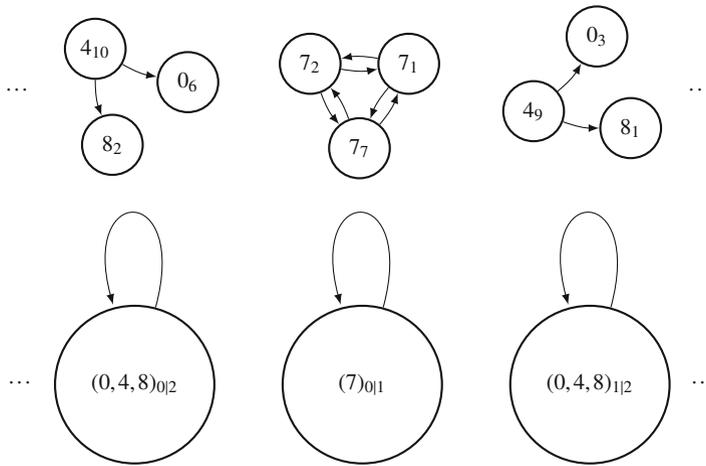


Fig. 4.5 A few Fuxian consonances from the strict digraph (above) and their enclosing weak components (below). Consonances $0_{2x} \leftarrow 4_{2x} \rightarrow 8_{2x}$ are connected in the underlying undirected graph, hence are all members of the same weak component $(0, 4, 8)_{0|2}$, on the left. In the weak digraph, the forbidden parallels of fifths 7_x appear as a complete subgraph—any step from a fifth to another is prohibited, whatever the cantus firmus. For the sake of legibility, loops of the strict digraph are not represented.

4.3.2.3 Strong Partitions

Connectivity in undirected graphs implies mutual reachability, a feature not shown by digraphs. As soon as one considers (directed) arrows, two directed paths are needed to join two different vertices, one for each direction.

Definition 4.10. Let $D = (V_D, A_D)$ be a strict digraph. The strong equivalence relates two vertices that can be mutually joined by directed paths.

$$\forall v_0, v_i \in V_D \quad v_i \sim_s v_0 \Leftrightarrow \exists v_0, \dots, v_i, \dots, v_n \in V_D : (v_j, v_{j+1}) \in A_D \text{ and } (v_n, v_0) \in A_D \tag{4.30}$$

for any $j \in \{0, \dots, n\}$.

Two vertices are strongly equivalent if they belong to a directed cycle, as shown in Figure 4.6. Any cycle of the digraph D is contained inside a strong component, turning the strong quotient digraph, also called *condensation* in the literature, into a *directed acyclic graph*.

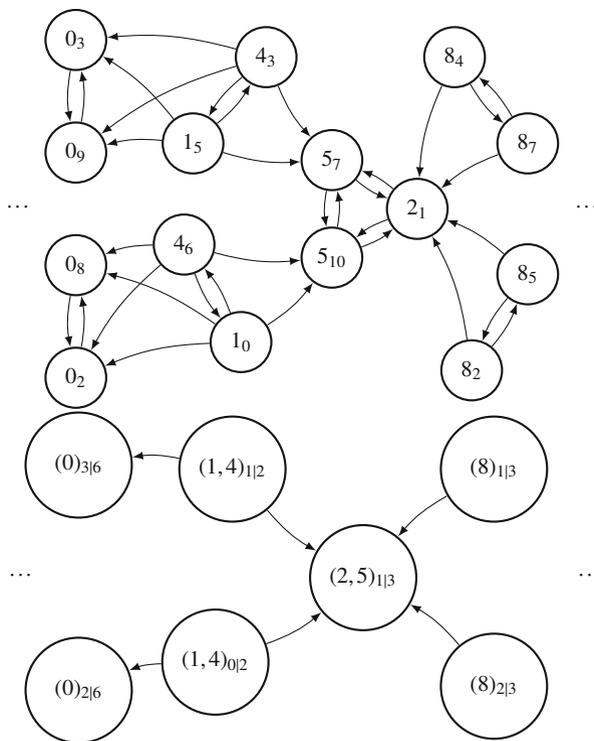


Fig. 4.6 Partial representation of the twelve-tone counterpoint world in class 75 of [57]. A few consonances of the strict digraph (above) and their enclosing strong components (below). Two directed cycles of opposite direction connect vertices 2_1 , 5_7 , and 5_{10} . On the other hand, 2_1 is reachable from 8_4 , but there is no way back, hence the two different components. For the sake of legibility, loops are not represented.

Consonances belonging to a same strong component seem to show much tighter bonds than just cycles. They form cliques, i.e., maximal complete subgraphs, in all worlds at least up to the twelve-note worlds.

Conjecture 4.1. *The strong components of a strict digraph related to a counterpoint world form cliques.*

The claim has been verified only for the worlds with lower octave division (up to $2k = 14$). In the general case, a proof or a counterexample is still needed for the higher microtonal cases.

Despite the theoretical interest of strong digraphs, we will turn to a fourth equivalence relation, which will be used extensively in this chapter and the next.

4.3.2.4 Homogeneous Partitions

Another way to identify vertices based on connectivity is to consider an identical neighborhood. Let $D = (V_D, A_D)$, we set

$$\begin{aligned} N_D^-(\xi) &:= \{\xi_0 \in V_D : (\xi_0, \xi) \in A_D\}, & \text{the in-neighborhood, and} \\ N_D^+(\xi) &:= \{\xi_1 \in V_D : (\xi, \xi_1) \in A_D\}, & \text{the out-neighborhood.} \end{aligned} \tag{4.31}$$

The in- and out-neighborhoods of (4.31) also include the consonance ξ , since the loops in strict digraphs connect every vertex to itself. Some authors exclude vertices from their neighborhoods, see Section A.2.

Definition 4.11. Let $D = (V_D, A_D)$ be a strict digraph. Two vertices are related by a homogeneous⁶ equivalence if they share exactly the same in- and out-neighborhoods.

$$\forall \xi_0, \xi_1 \in V_D, \quad \xi_1 \sim_h \xi_0 \Leftrightarrow \begin{cases} N_D^-(\xi_1) = N_D^-(\xi_0), \\ N_D^+(\xi_1) = N_D^+(\xi_0). \end{cases} \tag{4.32}$$

Because of the loops, a vertex is always equivalent to itself. Two different vertices will be related if and only if they form a 2-cycle. If there are n equivalent vertices, the pairwise 2-cycles will add up to form a complete subgraph K_n .

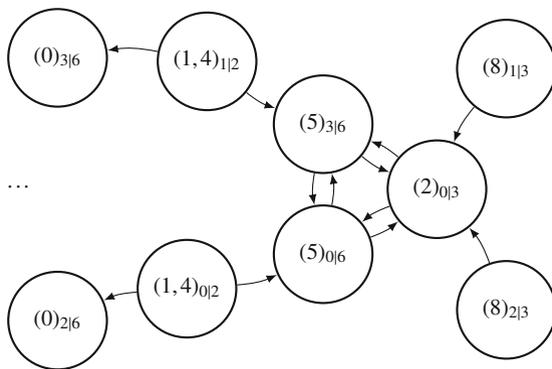


Fig. 4.7 Partial representation of the homogeneous digraph of the 12-note counterpoint world class 75 of [57]. Homogeneous components are the same as the strong components of Figure 4.6, except the central component $(2,5)_{1|3}$, which splits into three smaller components. Their neighborhoods differ: Vertex 2_1 is not connected to 4_3 , and 5_{10} is not connected to 8_7 , etc. For the sake of legibility, loops are omitted.

⁶ What we call homogeneous (finite) digraph/homogeneous digraphs in this book are not to be confused with the definition of a homogeneous infinite graph found in the literature, as in section 8.3 of [22].

4.3.2.5 Null Partitions

The last partition is the second trivial partition—the finest possible one, which identifies every vertex only with itself.

Definition 4.12. *Let $D = (V_D, A_D)$ be a strict digraph. Every vertex of D is null equivalent to itself and only to itself.*

$$\forall \xi_0, \xi_1 \in V_D, \quad \xi_1 \sim_n \xi_0 \Leftrightarrow \xi_1 = \xi_0 \quad (4.33)$$

The null partition is simply a replication of the vertex set, where each element is transformed into a set: $V := N \cup_{\xi \in V_D} \{\xi\}$.

The main purpose of the two trivial equivalences, null and full, is to have unified tools and notations for handling vertices, intermediate partitions, and strict digraphs.

4.3.2.6 Inclusion Hierarchy

The broader the partition, the simpler and more readable the quotient digraph. What appears at a broader level restricts what can happen at a finer level, a mechanism we will extensively use in Section 5.3.6. But for this to work, the different quotient digraphs have to be ordered according to the partition's granularity.

Definition 4.13. *Let $D = (V_D, A_D)$ be a strict digraph, and \sim_q, \sim_p be two equivalence relations defined on the vertex set. The quotient digraph $Q = (V_Q, A_Q)$ is said to be a subquotient digraph of the quotient digraph $P = (V_P, A_P)$ if, in terms of the associated partitions, \sim_q is a refinement of \sim_p . We write*

$$Q \subseteq_{\sim} P \Leftrightarrow \forall q \in V_Q, \exists p \in V_P : q \subseteq p. \quad (4.34)$$

The quotient inclusion defines a digraph homomorphism $\subseteq_{\sim} : V_Q \rightarrow V_P$ because we care only about the arrow sets and not their complement in the definition of quotient arrows (see Definition 4.6). Note that the quotient inclusion (4.34) is not to be confused with subdigraphs; the vertices of V_Q are not elements of V_P , but both are different subsets of the common universal set V_D .

The quotient inclusion defines always a linear order on four of the quotient digraphs defined in Section 4.3.2: $N \subseteq_{\sim} H \subseteq_{\sim} W \subseteq_{\sim} F$. The strong digraph S can be inserted into the chain above between the homogeneous and weak digraphs, as long as strong components form cliques. Homogeneous components are always complete subdigraphs, but it is not known yet if microtonal worlds without this property exist, see Conjecture 4.1.

If a strict morphism ϕ exists between the strict digraphs $D = (V_D, A_D)$ and $D' = (V_{D'}, A_{D'})$, the same equivalence relations defined on each vertex set fulfills the compatibility condition (4.26), and the following inclusion chain holds.

$$\begin{array}{ccccccccc}
 D & \xrightarrow{\pi_{\sim n}} & N & \xrightarrow{\subseteq_{\sim}} & H & \xrightarrow{\subseteq_{\sim}} & W & \xrightarrow{\subseteq_{\sim}} & F \\
 \phi \downarrow & & \phi_N \downarrow & & \phi_H \downarrow & & \phi_W \downarrow & & \phi_F \downarrow \\
 D' & \xrightarrow{\pi_{\sim n}} & N' & \xrightarrow{\subseteq_{\sim}} & H' & \xrightarrow{\subseteq_{\sim}} & W' & \xrightarrow{\subseteq_{\sim}} & F'
 \end{array} \tag{4.35}$$

Why the homogeneous morphism ϕ_H has to be injective will be explained in Section 4.3.3.5.⁷

4.3.3 Homogeneous Digraphs

A homogeneous digraph provides the best summary of a counterpoint world’s structure in the sense that it eliminates redundancy while preserving all the necessary details of the strict digraph’s structure. Each different connectivity profile appears as a homogeneous component, whose content tells which consonances show a similar behavior. Homogeneous arrows bundle strict arrows connecting similar profiles, thus expressing similar connections. Everything we need to know about a counterpoint world is described by its homogeneous digraph. If a homogeneous morphism connects two counterpoint worlds, then so does a strict morphism, as explained in Section 4.3.3.3.

As mentioned on pages 45 and 49, the Definitions 4.5 of a strict morphism and 4.11 of homogeneous equivalence may seem too restrictive at first glance. It is indeed possible to release some of the constraints imposed on the transformation rules and still produce valid counterpoints. However, the net benefit is a faithful transport of the contrapuntal structure between counterpoint worlds, as well as between the null (strict) and homogeneous levels of the quotient hierarchy, as noted in Section 4.3.3.2. The next sections explain why a strict digraph can be replaced by its simpler homogeneous equivalent without loss of information.

4.3.3.1 Homogeneous Components Form Maximal Preimages

A strict morphism does not need to be surjective, nor injective, but it has to preserve non-adjacency. Since a loop is attached to every target vertex, which forms a K_1 , any pair of vertices in the preimage must be adjacent, see Figure 4.8.

But not every complete subdigraph can be mapped to a single target vertex. Some are too big, such as those found in world 78 displayed in Figure 4.15: Four homogeneous components build a clique, the strong component $(1, 2)_{1|2}$. However, the homogeneous component $(2)_{1|2}$ is connected to $(0, 6)_{1|2}$ and not to any of the $(4)_{x|6}$ components that are only connected to the $(1)_{x|3}$ components. This strong component must be split into different parts, otherwise no homogeneous morphism and

⁷ Less interestingly, this is also the case for the trivial full morphism ϕ_F . It is impossible to not be injective when mapping a single element.

none of its related strict morphisms will be able to preserve the respective homogeneous and strict structures.

Homogeneous components hence correspond to the maximal subgraphs of strict digraphs that can be mapped to a single target vertex.

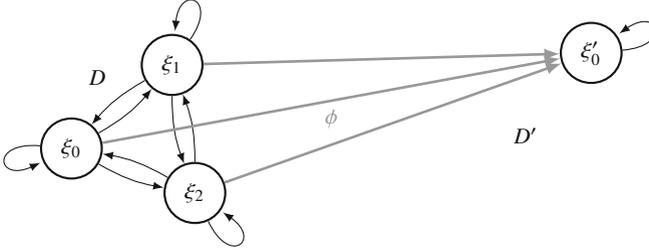


Fig. 4.8 If more than one source vertex ξ_i is mapped to the target vertex ξ'_0 , its preimage under the strict morphism ϕ must form a complete subgraph of the strict digraph D .

4.3.3.2 Homogeneous Morphisms Are Strict

A homogeneous morphism preserves both adjacency and non-adjacency of the source homogeneous digraph.

Lemma 4.2. *Let $\phi : V_D \rightarrow V_{D'}$ be a strict morphism. Then its associated homogeneous morphism $\phi_H : V_H \rightarrow V_{H'}$ is structure-preserving.*

Proof. By the definitions of homogeneous equivalence (4.31) and quotient arrows (4.25), two homogeneous components h_0 and h_1 are connected if and only if all of their included consonances are connected,

$$(h_0, h_1) \in A_H \Leftrightarrow (\xi_0, \xi_1) \in A_D, \quad (4.36)$$

for any $\xi_0 \in h_0$ and $\xi_1 \in h_1$. Therefore, given a homogeneous arrow $(h_0, h_1) \in A_H$, condition (4.36) implies that any pair of included consonances $(\xi_0, \xi_1) \in h_0 \times h_1$ are connected. Let ξ'_0 and ξ'_1 be the images of ξ_0 and ξ_1 , respectively, under ϕ . Because ϕ is strict,

$$(\xi_0, \xi_1) \in A_D \Rightarrow (\xi'_0, \xi'_1) \in A_{D'}. \quad (4.37)$$

Taking the converse implication in (4.36), it follows that

$$(\xi'_0, \xi'_1) \in A_{D'} \Rightarrow (h'_0, h'_1) \in A_{H'}, \quad (4.38)$$

where $\xi'_0 \in h'_0$ and $\xi'_1 \in h'_1$. The homogeneous morphism ϕ_H associated to the strict morphism ϕ is thus adjacency preserving: $(h_0, h_1) \in A_H \Rightarrow (h'_0, h'_1) \in A_{H'}$. A similar

reasoning applies for non-adjacency preservation. The structure-preserving character of ϕ propagates to ϕ_H by means of the homogeneous equivalence relation. \square

4.3.3.3 Homogeneous Morphisms Determine the Existence of Strict Morphisms

Applying the definitions of Section 4.3.2, it is straightforward to build every quotient morphism associated with a given strict morphism. On the other hand, the reverse construction might not always be feasible: Given an arbitrary function mapping quotient digraphs, can we build a compatible strict morphism?

Weak and strong components are too big to allow that. They encapsulate too much detail of the original contrapuntal structure, and simply preserving the weak or strong structure is not enough to enable the construction of a strict morphism. For instance, each of the three twelve-tone worlds shown in Figures 4.13, 4.14, and 4.15 form a single weak component. Any trivial function would preserve the weak structure. However, Figure 5.14 shows that no strict morphism exists between any of them. Structural incompatibilities that prevent the existence of a strict morphism might not appear before the homogeneous level is studied. Not only is it necessary to work at this level, but it is also sufficient, as the following lemma shows.

Lemma 4.3. *Let $\phi_H : V_H \rightarrow V_{H'}$ be any injective function preserving the homogeneous structure. Then any function $\phi : V_D \rightarrow V_{D'}$ satisfying*

$$\forall \xi \in V_D, \quad \xi \in h \Rightarrow \phi(\xi) \in \phi_H(h) \quad (4.39)$$

is a strict morphism.

Proof. If condition (4.39) holds, simply replacing inclusions by the canonical projections yields the commutative diagram (4.26), so that the compatibility of ϕ with ϕ_H is guaranteed. We now have to prove that such a ϕ is indeed a strict morphism, i.e., that it preserves the structure of D .

Let $h_0 \mapsto h'_0$ and $h_1 \mapsto h'_1$ be any pair of homogeneous mappings under ϕ_H . Using the same reasoning as in Section 4.3.3.2, we see that the structure of D is propagated through the entire chain:

$$\begin{aligned} & (\xi_0, \xi_1) \in A_D \\ \Leftrightarrow & (h_0, h_1) \in A_H \\ \Leftrightarrow & (h'_0, h'_1) \in A_{H'} \\ \Leftrightarrow & (\xi'_0, \xi'_1) \in A_{D'}. \end{aligned} \quad (4.40)$$

\square

Homogeneous morphisms do not only allow any mappings inside components, they also constrain mappings between components, as shown in the two following sections.

4.3.3.4 Homogeneous Components Do Not Split

Adjacency-preservation of ϕ guarantees that homogeneous morphisms are well-defined functions, i.e., they cannot map a single source component to different target components.

Given a homogeneous mapping, consonances from the source component can only be mapped to consonances contained inside a single target component, see an illustration in Figure 4.9. A strict mapping cannot leave the ‘channel’ provided by the homogeneous mapping, if other mappings already follow this route. A homogeneous mapping keeps strict mappings tightly bound.

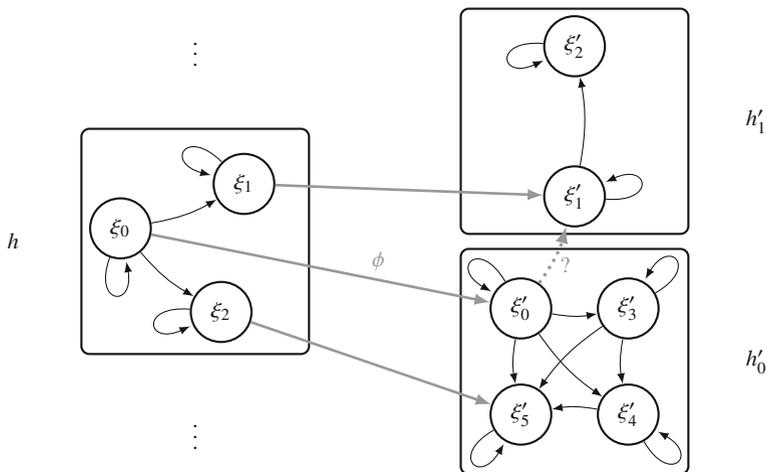


Fig. 4.9 A homogeneous morphism must be a well-defined function, i.e., all vertices inside a source component must be mapped to a unique target component. If this is not the case, we end up in a situation like the one illustrated here, where vertex ξ'_1 belongs to a component h'_1 different from h'_0 , which contains other images. The function ϕ does not preserve the structure of D since the connection between ξ_0 and ξ_1 does not exist between ξ'_0 and ξ'_1 .

4.3.3.5 Homogeneous Morphisms Do Not Merge Components

Non-adjacency preservation of ϕ , i.e., $\phi(A_{D^c}) \subseteq A_{D^c}$, ensures that ϕ_H is injective. Source consonances belonging to different components cannot be mapped to consonances in the same target component, as shown in Figure 4.10.

A homogeneous arrow bundles strict arrows inside strict digraphs. The same phenomenon happens with mappings between source and target digraphs. A homogeneous mapping bundles and summarizes strict mappings connecting its source and

target components. A strict mapping cannot enter or leave the source-target combination; it has to follow the route shown by the homogeneous mapping.

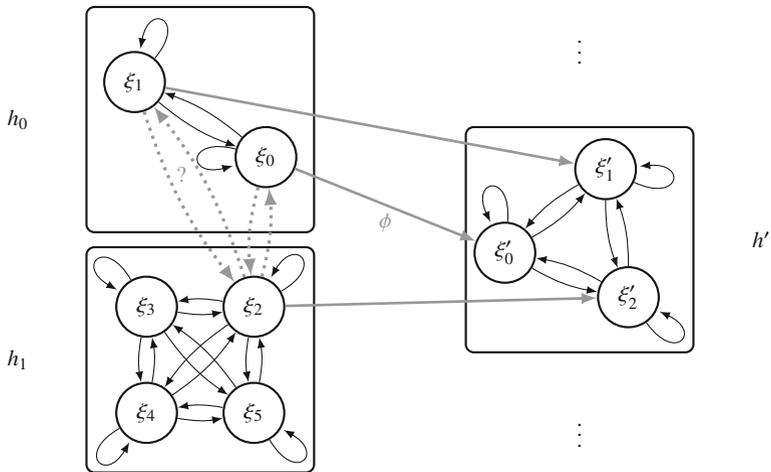


Fig. 4.10 Two images ξ'_0 and ξ'_1 lying in the same target component h' share the same neighborhood. If the preimages ξ_0 and ξ_1 are not equivalent, they are not bound by the same structure as the images, and this implies that some arrow is missing in the source digraph, contradicting the non-adjacency preservation.

4.3.3.6 Gallery of Homogeneous Digraphs

Having emphasized the importance of homogeneous digraphs in the previous section, we end this chapter on digraphs with illustrations of the contrapuntal structure of the six twelve-tone worlds.

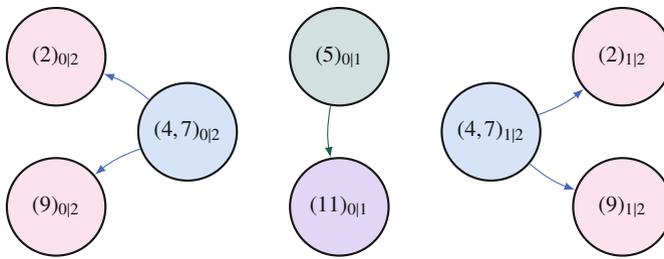


Fig. 4.11 Homogeneous graph of the Ionian world, or class number 64 in [57]. All numbers denote residue classes modulo 12. Different colors represent different orbits under the automorphism group. For the sake of legibility, loops are not represented.

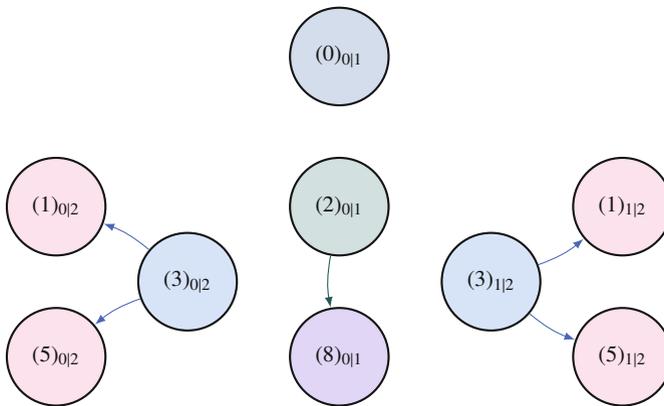


Fig. 4.12 Homogeneous graph of the twelve-tone world having class number 68 in [57]. All numbers denote residue classes modulo 12. Different colors represent different orbits under the automorphism group. For the sake of legibility, loops are not represented.

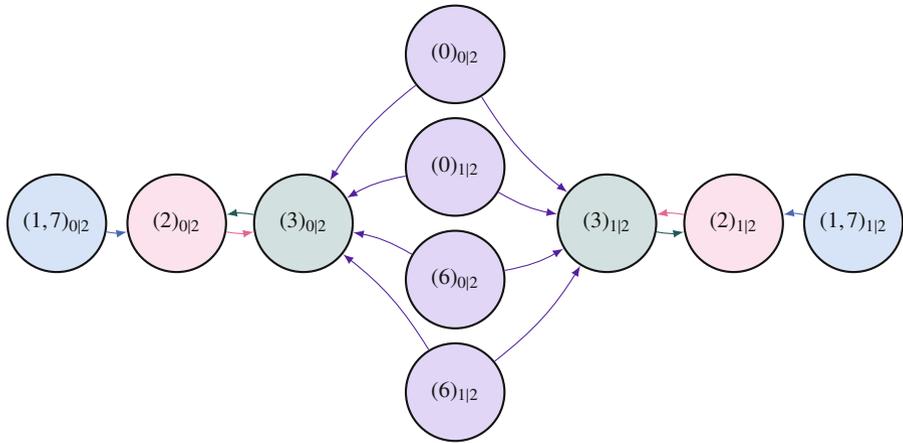


Fig. 4.13 Homogeneous graph of the twelve-tone world having class number 71 in [57]. All numbers denote residue classes modulo 12. Different colors represent different orbits under the automorphism group. For the sake of legibility, loops are not represented.

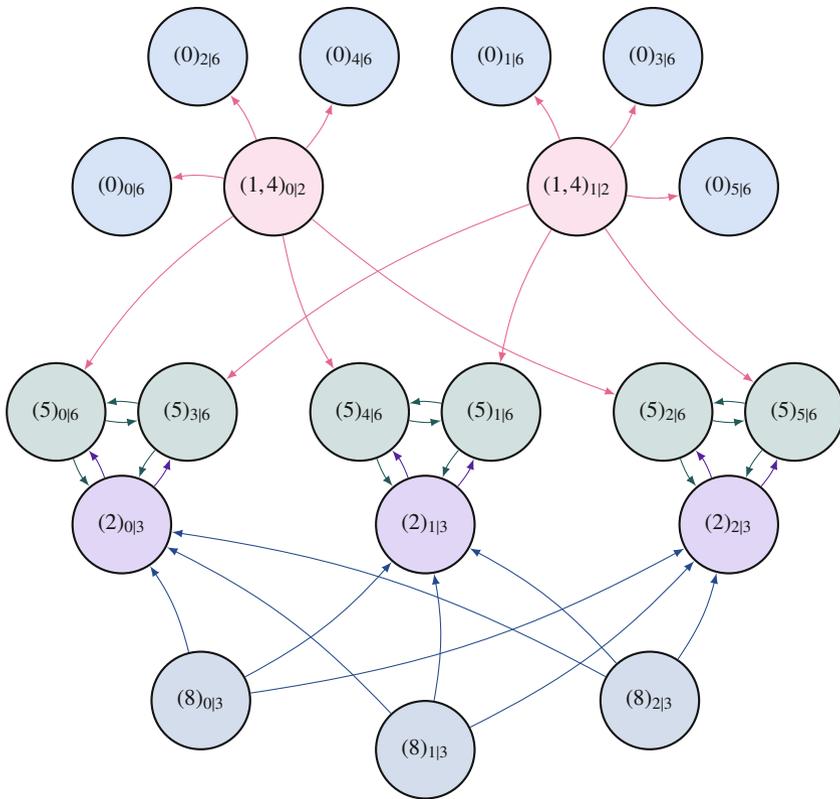


Fig. 4.14 Homogeneous graph of the twelve-tone world having class number 75 in [57]. All numbers denote residue classes modulo 12. Different colors represent different orbits under the automorphism group. For the sake of legibility, loops are not represented.

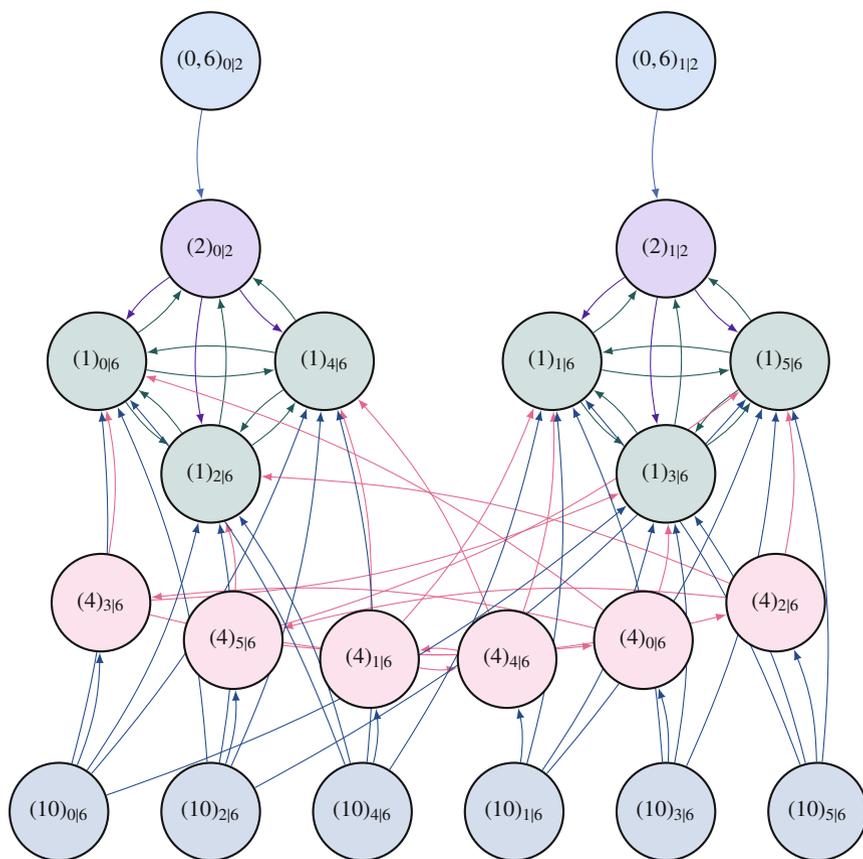


Fig. 4.15 Homogeneous graph of the twelve-tone world having class number 78 in [57]. All numbers denote residue classes modulo 12. Different colors represent different orbits under the automorphism group. For the sake of legibility, loops are not represented.

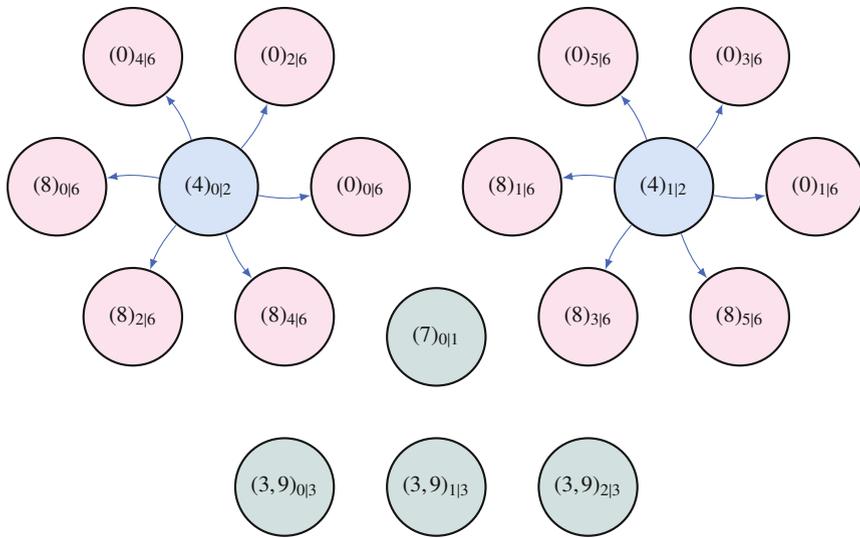


Fig. 4.16 Homogeneous graph of the Fuxian world, or class number 82 in [57]. All numbers denote residue classes modulo 12. Different colors represent different orbits under the automorphism group. For the sake of legibility, loops are not represented.

Chapter 5

Morphism Enumeration

Is it possible to take a piece of music, composed in compliance with the rules of a given counterpoint world, and transform it into a new counterpoint conforming to a different set of rules? Could we, for example, take a Fuxian counterpoint and alter its pitches so that it becomes a valid Ionian composition (see Section 3.4)? We stated in Definitions 4.2 and 4.5 that the specifications such a transformation must meet, but we said nothing about their existence. Whether a morphism can be constructed or not depends on the choice of source and target worlds, and on the source counterpoint. This chapter examines the existence question and describes a recipe for finding all possible counterpoint world morphisms between two given worlds or, if none can be found, for proving that they do not exist.

Translated into the language of graph theory, searching for counterpoint world morphisms boils down to searching for strict morphisms. More precisely, the question we have to answer is the following: Given a pair of homogeneous digraphs H and H' , can we embed H into H' , i.e., can we find a subgraph of H' isomorphic to H ? This kind of problem is an instance of the more general *subgraph isomorphism* problem (SGI), also called *graph matching*. It shows up in many areas, spanning from computer vision, or pattern recognition [17], to information retrieval from chemical structure databases [25].

Subgraph isomorphism is a famous example of an NP-complete problem. It is mentioned in the classical book of Garey and Johnson [31, p. 64]. While the validity of a proposed solution can be checked quickly (i.e., in polynomial time, with a worst-case growth of $W(n^2)$ if the source digraph is of order n), it is much harder to find an actual solution. There is no known procedure for constructing a solution in polynomial time. We will always face the risk that practical algorithms performing well in average situations show exponential growth on unfavorable cases.

Algorithms for solving the SGI problem fall into two categories: whether we tolerate approximate solutions or need exact ones. Loosening constraints and allowing suboptimal solutions may be one way to speed up the search process.

In the case of counterpoints and computer-assisted composition, a digraph matching must be *exact*. The faithful preservation of rules, as described by the digraph's

structure, is mandatory; no approximate method can be used.¹ Another point to keep in mind is that we are not only verifying the existence of a morphism (detection), we are also looking for *every* possible solution (enumeration). As will be explained in Section 6.2.3, the composer should be able to navigate among all morphisms easily, basing his or her choice on sole creative considerations, not on technical hindering.

Another solution is to restrict the class of input graphs to certain types of graphs with particular structures. For example, linear growth in execution time with regards to number of vertices has been reported when dealing with trees, see theorem 3.3 on page 85 in [7].

Algorithms capable of handling arbitrary digraphs and finding exact solutions are far less numerous than the approximate or specialized ones. The most famous example has been developed by Ullmann [84]. It relies on local considerations such as neighborhoods of vertices, a strategy also adopted by the more recent *VF2* [19]. An example of a global approach is given by [96], who borrowed the partition-refinement technique from the related *graph isomorphism* (GI) problem-solving techniques.

The most popular algorithm for GI based on the latter approach is called *nauty*. It relies on the enumeration of internal symmetries, or automorphisms, and a recent update [64] makes the following suggestion:

For example, sets of vertices with the same neighborhoods ought to be replaced by single vertices with a color that encodes the multiplicity.

This is exactly what homogeneous graphs are supposed to do and the reason why they are used extensively in this chapter. The complexity of GI is still an open question, see, for example, [47]. The fact that the image of the isomorphism is known in advance (i.e., the entire target digraph) allows the computation of certificates for isomorphism detection. This is not possible with subgraphs, where the image is not known before the entire morphism has been built. Nevertheless, automorphisms are still useful for guiding the morphism construction.

Broadly speaking, the algorithm can be specified as follows:

| | |
|---------------|--------------------------------------------------------------------------------------|
| <i>Task</i> | Find every counterpoint world morphism between two counterpoint worlds. |
| <i>Input</i> | Source (local or global) and target (global) counterpoint worlds. |
| <i>Output</i> | Set of all possible counterpoint world morphisms (empty if worlds are incompatible). |

A systematic exploration of the combinatorial space will find every exact solution to the SGI problem, but at the cost of exponential growth. Section 5.1 explains how this method works. Three principles can be applied to reduce the amount of data to process, hopefully enabling reasonable execution times:

1. avoid redundancy, see Section 5.2.1;
2. avoid large combinations, see Section 5.2.2;
3. avoid unnecessary work, see Section 5.2.3.

¹ Unless one dreams up more-tolerant fuzzy rules and matching tolerant deviations.

Throughout this chapter, we will address the graph-theoretical equivalent of this problem, namely finding strict morphisms between two given strict digraphs. The discussion starting on page 49 explains why we can work with strict morphisms instead of counterpoint world morphisms.

5.1 Backtracking

In order to collect all existing morphisms, we apply a technique called *backtracking*.² The idea is to construct a morphism $\phi : V_D \rightarrow V_{D'}$ step-by-step by successively adding individual vertex mappings $v \mapsto v'$ and to systematically explore this space of combinations. In this context, it is worth thinking of a function ϕ as a set-theoretic *relation*, i.e.,

$$\phi \subseteq V_D \times V_{D'}, \quad (5.1)$$

where a mapping, seen as an ordered pair (v, v') , is also an element of the subset ϕ .

Morphisms resulting from the combination of mappings: Each node of the search tree, like the one depicted in Figure 5.1, represents one step in a possible construction sequence. Should we perform this exercise on strict digraphs, we would get

$$\begin{aligned} \phi_i^{(0)} &= \emptyset, \\ \phi_i^{(1)} &= \{(v_1, v'_1)\}, \\ \phi_i^{(2)} &= \{(v_1, v'_1), (v_2, v'_2)\}, \\ &\vdots \\ \phi_i^{(l)} &= \{(v_1, v'_1), (v_2, v'_2), \dots, (v_l, v'_l)\}, \\ &\vdots \\ \phi_i^{(2k)} &= \phi_i, \end{aligned} \quad (5.2)$$

where l indicates the level of the tree's node. Each sequence starts with an empty set at level $l = 0$, and reaches completion, if possible, at level $l = 2k$, the order of the digraph.

As long as we are taking only the combinatorial aspect into account, we are facing an exponential growth issue. There are

$$|V_{D'}|^{|V_D|}$$

possible functions of vertex sets. Each candidate function must be verified for structure preservation, yielding $|V_D|^2$ comparisons of connectivity between all pairs of source vertices and their respective images. In the case of the traditional twelve-

² See for example [67, chapter 5], which gives an introduction to the subject.

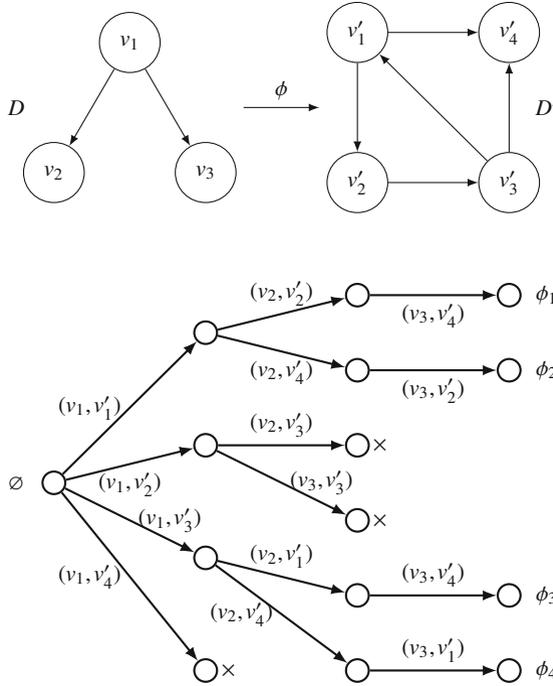


Fig. 5.1 The systematic search for all four ($i = 1, \dots, 4$) possible morphisms $\phi_i : D \rightarrow D'$ is carried out by successively adding individual mappings of a vertex v in D to a vertex v' in D' . Each node of the search tree, or state space tree, represents a state, in this case a partially constructed morphism. Arrows between states indicate the addition of a new mapping $v_i \mapsto v'_i$ to the existing collection. Obviously invalid mappings, such as those not preserving in- or out-degrees, are not shown. The node on the far left is the root of the search tree: The procedure starts with an empty morphism at level $l = 0$. Leaves on the far right, located at level $l = |V_D| = 3$ are complete valid morphisms, marked as ϕ_i . Leaves located at intermediate levels ($0 < l < 3$) are dead ends that do not lead to the complete construction of a morphism.

tone worlds, this means more than $5.3 \cdot 10^{133}$ configurations to build and up to 5,148 adjacencies to test each time, while most of the combinations do not yield valid morphisms but dead ends.

Nevertheless, blind combination testing is not the end of the story. A backtracking approach yields practical results only if dead ends can be avoided. If we can identify a node in the search tree as not leading to a valid solution, we will save computation time by stopping any further investigation along this branch. This happens, for example, if a partially constructed morphism contains a pair of mappings violating the structure. Adding any further mappings will not correct this problem. In this case, the procedure removes the last addition, which caused the problem, and tries adding the next mapping. In the search tree, this corresponds to a move back to the parent node, hence the name *backtracking*. It will resume searching from the next state by adding the next sibling, if there is one available, or move back to the

parent, the grandparent, etc., looking for the next child nodes in the search tree. Such *non-promising* nodes, where the procedure backtracks, are marked with a cross in Figure 5.1.

For our purpose, we will not use the backtracking approach on entire strict digraphs, only on subsets of homogeneous digraphs. This is part of the size-reduction strategy explained in Section 5.2.2.

5.2 Reducing the Problem Size

Applying brute-force backtracking, as presented in Section 5.1, to the construction of entire strict morphisms is intractable. The size of the data and the number of operations must first be reduced—for example by processing multiple vertices at once and by limiting the range of combinations to inspect.

5.2.1 Avoiding Redundancy

The structure of strict digraphs shows a lot of redundancy, which can be expressed by equivalence relations. We can thus take advantage of homogeneous digraphs, whose order and size are lower than their strict counterparts. Table 5.1 shows a few examples comparing quotient and strict digraphs - note the significant differences in size in some cases.

There are two reasons why quotienting strict digraphs significantly reduces their size. First, according to Conjecture 4.1, strong components usually form cliques in strict digraphs. Second, under these circumstances, loops force homogeneous components to be subgraphs of strong components. An arbitrary digraph may not have loops, and its strong components may consist only of cycles. Thus there might be very few vertices sharing an identical neighborhood, and without sufficiently dense connections, such a digraph may yield a homogeneous digraph that is an exact replica of itself.

A benefit of using homogeneous structures, rather than strict ones, is that there are less of them, so that they can be enumerated. Once a homogeneous morphism has been found, generating an associated strict morphism is a simple task of combining strict mappings compatible with the homogeneous mappings. This process will always yield a valid strict morphism and can be carried out on-the-fly, so we do not need to compute every strict morphism in advance.

Remember that a homogeneous mapping $h \mapsto h'$ guides and filters possible mappings at the null (strict) level:

1. Target consonances in h' get locked by the homogeneous mapping. They become unavailable for any mapping originating from a source component different from h .

2. Source consonances in h get locked by the homogeneous mapping. Once a consonance in h has been mapped to h' , all components different from h' become unavailable as targets for other mappings originating from h .
3. There is total freedom of mapping consonances inside the homogeneous mapping. For any pair of consonances $\xi_0, \xi_1 \in h$ and $\xi'_0, \xi'_1 \in h'$, we have $(\xi_0, \xi_1) \in A_D$ and $(\xi'_0, \xi'_1) \in A_{D'}$. Therefore, there is no risk of violating the structure, since h and h' both form complete subgraphs. All mappings are compatible and any consonance in h can be mapped to any consonance in h' .

Table 5.1 Order $|V|$ and size $|A|$ of strict digraphs D and their associated homogeneous digraphs H , according to octave division $2k$ and counterpoint world index Δ . The reduction in number of vertices and arrows is given by the ratios ρ_V and ρ_A , respectively.

| $2k$ | Δ | $ V_D $ | $ V_H $ | ρ_V | $ A_D $ | $ A_H $ | ρ_A |
|------|----------|---------|---------|----------|---------|---------|----------|
| 2 | 2 | 2 | 1 | 50% | 2 | 1 | 50% |
| 4 | - | - | - | - | - | - | - |
| 6 | 7 | 18 | 5 | 28% | 66 | 5 | 8% |
| 8 | 13 | 32 | 8 | 25% | 256 | 16 | 6% |
| | 16 | | 8 | 25% | 256 | 16 | 6% |
| 10 | 33 | 50 | 4 | 8% | 750 | 5 | 1% |
| | 36 | | 12 | 24% | 510 | 17 | 3% |
| | 39 | | 4 | 8% | 850 | 4 | 0% |
| | 44 | | 12 | 24% | 510 | 17 | 3% |
| | 45 | | 4 | 8% | 850 | 4 | 0% |
| | 46 | | 4 | 8% | 750 | 5 | 1% |
| 12 | 64 | 72 | 8 | 11% | 1,152 | 13 | 1% |
| | 68 | | 9 | 13% | 936 | 14 | 1% |
| | 71 | | 10 | 14% | 1,152 | 24 | 2% |
| | 75 | | 20 | 28% | 984 | 59 | 6% |
| | 78 | | 22 | 31% | 960 | 96 | 10% |
| | 82 | | 18 | 25% | 600 | 30 | 5% |

As mentioned in Section 4.3.3, homogeneous digraphs provide a summary of the contrapuntal structure. They yield basic mapping constraints, primarily restrictions on in- and out-degrees, that can serve to detect the absence of morphisms. For example, among the twelve-tone worlds, world Δ_{68} , depicted in Figure 4.12, cannot be mapped into the Ionian world Δ_{64} of Figure 4.11 because of an excess of components: There is no space left for the lonely component $(0)_{0|1}$. In simple cases, visual inspection of the homogeneous digraphs can let one judge if a morphism exists or not. The existence of a morphism is harder to tell on strict digraphs; you can compare the homogeneous digraphs in Figures 4.16 and 4.11 with their strict structures in C.6 and C.1.

Quotient structures can be organized into trees whose parent-child arrows reflect inclusion relations (defined in Section 4.3.2.6). This sort of “meta”-graph, or digraph of digraphs, allows easy navigation between full, weak, homogeneous, and

null levels. [Table 5.2](#) lists the ordering and notations used throughout this chapter to designate the different levels of quotient structures.

Definition 5.1. *The component tree $T_Q = (V_{T_Q}, A_{T_Q})$ of a strict digraph D is the tree whose nodes are the quotient components belonging to each of the levels listed in [Table 5.2](#),*

$$V_{T_Q} := \{q \in V_{Q^{(l)}} : l = 0, 1, 2, 3\}. \quad (5.3)$$

Arrows represent successive containments, ordering components from the broadest to the finest level:

$$A_{T_Q} := \{(q^{(l)}, q^{(l+1)}) \in V_{Q^{(l)}} \times V_{Q^{(l+1)}} : q^{(l+1)} \subseteq q^{(l)}, l \in \{0, 1, 2\}\}. \quad (5.4)$$

The root node is the full component f , representing the entire strict digraph. Each leaf is a set containing a single consonance. For practical reasons, the strong level is skipped, since in many cases its partition is very similar—if not identical—to the homogeneous partition.

Quotient mappings can be organized according to the same hierarchy governing the source and target components. This defines a second tree structure.

Definition 5.2. *The mapping tree T_ϕ of a pair of strict digraphs D and D' is a directed tree whose nodes are mappings belonging to a quotient morphism. For each level $l = 0, 1, 2, 3$, let $V_{T_\phi}^{(l)}$ be the quotient mappings of level l :*

$$V_{T_\phi}^{(l)} := \{(q, q') \in V_{Q^{(l)}} \times V_{Q'^{(l)}} : \exists \phi_Q : V_{Q^{(l)}} \rightarrow V_{Q'^{(l)}} \text{ such that } (q, q') \in \phi_Q\}. \quad (5.5)$$

The entire vertex set is the disjoint union $V_{T_\phi} := \bigsqcup_{l=0,1,2,3} V_{T_\phi}^{(l)}$. Arrows point to children picked up from the finer predecessors in the inclusion chain (4.35). For any $l \in \{0, 1, 2\}$,

$$A_{T_\phi} := \{((q^{(l)}, q'^{(l)}), (q^{(l+1)}, q'^{(l+1)})) \in V_{T_\phi}^{(l)} \times V_{T_\phi}^{(l+1)} : q^{(l+1)} \subseteq q^{(l)}, q'^{(l+1)} \subseteq q'^{(l)}\}. \quad (5.6)$$

The contributions of components alone to problem size reduction (ρ_V in [Table 5.1](#)) may not be impressive. However, as soon as components are combined into mappings and morphisms, these modest reductions get multiplied and the size of the combinatorial space shrinks dramatically. For example, if h and h' were to contain 12 consonances each, there would be 12^{12} different combinations of mappings between these two components, or approximately $9 \cdot 10^{12}$ combinations corresponding to only one homogeneous mapping. But cutting orders and sizes of digraphs by a half or a factor ten will not prevent combinatorial explosion. Therefore, it is essential to circumscribe the domain on which we perform combinations.

5.2.2 Limiting Combinations

If we look at entire morphisms, resulting from the systematic combination of mappings, we will collect every leaf in the search tree, but the number of homogeneous morphisms found may exceed the storage capacity of today's computers. Keeping track of entire morphisms is not feasible: They incorporate far too many internal combinations.

On the other hand, if we look at isolated mappings, their number is much more limited. However, not every combination of mappings yields a valid morphism. According to Definition 5.2, a mapping is valid if it is part of a morphism. Therefore, two different mappings (h_1, h'_1) and (h_2, h'_2) could be incompatible in the sense that among every morphism ϕ_H containing (h_1, h'_1) , none contains (h_2, h'_2) , and vice versa. Both mappings never belong to the same morphism simultaneously, i.e., there is no ϕ_H such that $(h_1, h'_1) \in \phi_H$ and $(h_2, h'_2) \in \phi_H$, forbidding any morphism construction. Keeping track of isolated mappings, though feasible, does not provide a safeguard against invalid configurations. The perimeter is too small to contain any information about their surrounding structure.

To avoid the two aforementioned problems, we have to find a combinatorial compromise, i.e., domains of intermediate size on which to keep track of combinations. Storing a limited number of building blocks instead of complete combinations is another way to reduce the problem's size. The domains must be large enough to encapsulate the structure information necessary to avoid conflicts, and small enough to keep the number of internal combinations reasonably low. This results in partial combinations operating as building blocks whose combinations

1. are safe (one does not have to worry about possible incompatibilities),
2. cover the entire spectrum of valid morphisms.

The idea behind this strategy is to pre-calculate all valid combinations inside a limited area, identifying and removing possible conflicts—providing total freedom to assemble these pieces together into a morphism.

So what is an appropriate size for a safe domain? Remember that homogeneous morphisms have to preserve structure and be injective. Since no connections exist between two weak components, structure and injectivity are always preserved, and homogeneous mappings can be treated independently. As long as they belong to different weak components, there is no need to verify anything. However, combinations inside the same weak mapping must be checked carefully.

Inside a weak component, all vertices are somehow connected. Isolated weak components are the smallest perimeter for investigating combinations and their possible conflicts. But they may not be sufficiently large. Weak morphisms need not be injective; it is possible that several components are mapped to the same target and still yield a valid homogeneous morphism. This happens when the target is large enough to hold disconnected images of each source component, a situation frequently encountered in practice because counterpoints avoid forbidden steps. Their local counterpoint worlds yield almost discrete digraphs, as can be seen from [Figure 5.7](#). In this case, we cannot consider weak mappings in isolation. We

have to inspect simultaneous combinations involving a collection of weak mappings $\{(w_i, w')\}_{i \in I}$, all pointing to the same target component w' . Otherwise we may miss some possible combinations, and therefore morphisms.

It is the second task of the algorithm, beside populating and filtering the quotient mapping tree T_ϕ , to identify which weak mappings can simultaneously point to the same target and keep track of all corresponding homogeneous combinations that yield a local morphism. We will call them *local combinations*. Instead of giving the complete list of homogeneous morphisms, the algorithm will output a limited number of such local homogeneous combinations. The output of the algorithm will thus be an associative set M_ϕ telling which homogeneous combinations are possible under a given collection of weak mappings.

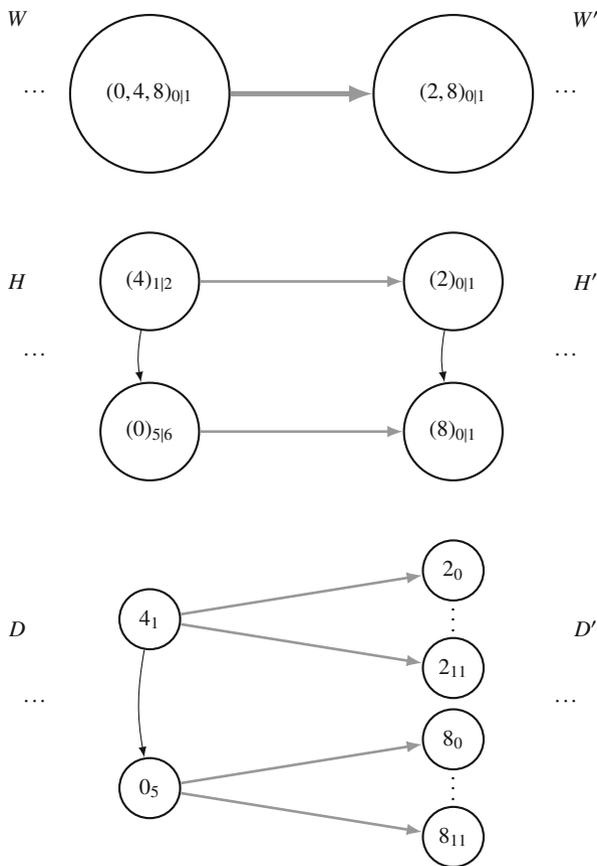


Fig. 5.2 Local homogeneous combinations available under the weak mapping $(0, 4, 8)_{0|1} \mapsto (2, 8)_{0|1}$. There is only one possible combination at the homogeneous level, but $12 \cdot 12$ ways to send 4_1 to 2_x and 0_5 to 8_x .

5.2.3 Pruning the Search Tree

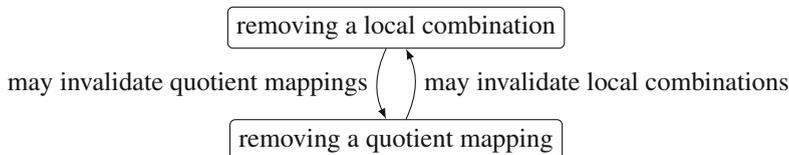
The third strategy for limiting the growth of the combinatorial search is to avoid unnecessary work. A series of conditions can be verified at each node of the search tree. If the test fails, the node is marked as *not promising*, because further combinations will not lead to a valid morphism. Detecting such dead ends as early as possible will prevent the algorithm from computing useless combinations along entire branches of the tree.

A couple of easily verifiable *necessary* conditions exist that detect mappings that can never belong to a valid morphism and hence must be rejected. To be sure that the remaining mappings indeed belong to a morphism, we need *sufficient* conditions. The problem is that no simple criterion is known for detecting invalid mappings that passed the previous tests. We are left over facing the circularity of the definition of a *valid* homogeneous mapping: A valid mapping is a mapping that belongs to a valid morphism, which is built by assembling valid mappings (see Definition 5.2). An iterative trial and error strategy must be applied: Start with a selection of homogeneous mappings that seem to be reasonably good candidates, try to assemble them into a homogeneous morphism, and repeatedly discard mappings that eventually cause problems. In order to verify the validity of mappings and morphisms, trying out combinations at some point is unavoidable. Nevertheless, we can limit the work by restricting the test to the smallest partial combinations possible, as discussed in Section 5.2.2.

Quotient mappings are stored in T_ϕ and are part of combinations stored in M_ϕ . Removing a mapping is not an isolated event and may affect other mappings as well.

1. Downward: A removed mapping automatically invalidates its descendant mappings, i.e., all mappings it contains at finer levels.
2. Outward: It automatically invalidates any local combination it is part of.
3. Upward: It may invalidate its parent mapping if there are not enough valid children left (Section 5.2.3.4) or no more valid local combinations (Section 5.2.3.5).

Every invalidation can trigger new removals. Removing a parent mapping together with all local combinations containing the removed mapping may then invalidate the grandparent and trigger an upward cascade of removals across the chain of ancestors until the root node. An empty T_ϕ means no morphism.



Filtering of homogeneous mappings and local combinations takes place in two phases. A preliminary screening is based on necessary conditions. Simple homogeneous digraph parameters help eliminate obviously invalid candidates. Next, the collected mappings are filtered iteratively. Verifications occur at weak and homogeneous levels, involving simultaneous combinations of several homogeneous map-

pings. This second step is time-consuming, so it is essential to eliminate as many invalid mappings as possible during the preliminary phase to reduce the amount of work to be done during the subsequent passes.

The search tree is like a map for combinatorial exploration, registering and displaying routes to every morphism. This information provides a tool for classifying different morphisms according to their similarity. Intermediate nodes are partially constructed morphisms, and they can indicate to what extent descendant morphisms share identical mappings. The higher the level of a common ancestor node, the more individual mappings are shared in the common branch, and the higher the degree of similarity between two different morphisms. On the other hand, the tree contains every path leading to a leaf, i.e., a valid morphism, so that it can be used to navigate to each of them. But the search tree can be used safely for such purposes only once we get rid of all invalid branches and dead ends.

Therefore, the main task of the algorithm is to select only those mappings and their combinations that are really constituent parts of morphisms. Combinations of mappings will cause trouble if they do not preserve digraph structures, if they violate injectivity at the homogeneous level, or if they split components. The next sections give a list of criteria used for filtering the search tree.

5.2.3.1 Homogeneous Embedding Condition

Inspection of the most basic digraph parameters can reveal if a pair of counterpoint worlds may not yield a morphism³. A first condition is the injectivity of homogeneous morphisms. The target homogeneous digraph must be big enough to hold the source components and arrows. For example, the 22 vertices and 96 arrows of the Fuxian world shown in [Figure 4.16](#) exceed the 8 vertices and 13 arrows of the Ionian world, [Figure 4.11](#). In terms of digraph order and size,

$$\exists \phi : V_D \rightarrow V_{D'} \Rightarrow \begin{cases} |V_H| \leq |V_{H'}|, \\ |A_H| \leq |A_{H'}|. \end{cases} \quad (5.7)$$

If the condition cannot be satisfied, the full mapping must be excluded. The same test can serve for excluding a weak mapping (w, w') :

$$\exists \phi_W : V_W \rightarrow V_{W'}, (w, w') \in \phi_W \Rightarrow \begin{cases} |V_{H[w]}| \leq |V_{H'[w']}|, \\ |A_{H[w]}| \leq |A_{H'[w']}|. \end{cases} \quad (5.8)$$

If the condition cannot be satisfied, the weak mapping must be excluded.

³ Section 2.5 of [33] lists a few such criteria based on graph invariants for assessing the existence of isomorphisms between simple graphs.

5.2.3.2 Homogeneous Component Embedding Condition

The same reasoning applies locally. As a consequence of the injectivity of ϕ_H , a target vertex h' must have enough neighbors to hold the neighborhood of the source vertex h , as shown in Figure 5.3. Incoming and outgoing arrows must be tested separately. For every $h \in V_H$ and $h' = \phi_H(h)$,

$$\exists \phi : V_D \rightarrow V_{D'} \Rightarrow \begin{cases} d_H^-(h) \leq d_{H'}^-(h'), \\ d_H^+(h) \leq d_{H'}^+(h'). \end{cases} \tag{5.9}$$

If the condition cannot be satisfied, the homogeneous mapping must be excluded. The verification could be strengthened by extending the comparison domain to neighbors of neighbors, and so on. This approach has been adopted by Cordella et al., who report a significant performance increase [19]. For the sake of simplicity, we limit the process to direct neighbors only.

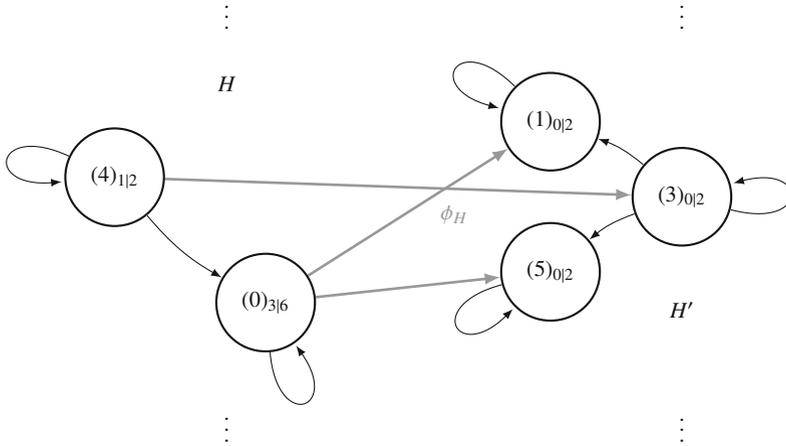


Fig. 5.3 Partial representation of the homogeneous digraphs H and H' . Component $(4)_{1|2}$, whose in-degree is 1 and out-degree 2, can only be mapped to component $3_{0|2}$, which possesses the same degrees. There are more choices for component $(0)_{3|6}$: $(1)_{0|2}$ and $(5)_{0|2}$ have the same degrees and are both valid candidates for mapping.

5.2.3.3 Homogeneous Complement Embedding Condition

Because of the preservation of non-adjacency, homogeneous components located outside the neighborhood of the source component cannot be mapped to the neighborhood of the target component. Once a vertex has been mapped, there should be enough space left in the target to hold the rest of the vertices, as illustrated in

Figure 5.4. For any homogeneous mapping (h, h') ,

$$\exists \phi : V_D \rightarrow V_{D'} \Rightarrow \begin{cases} |V_H[V_H \setminus N_H(h)]| \leq |V_{H'}[V_{H'} \setminus N_{H'}(h')]| \\ |A_H[V_H \setminus N_H(h)]| \leq |A_{H'}[V_{H'} \setminus N_{H'}(h')]| \end{cases} \quad (5.10)$$

If the condition cannot be satisfied, the homogeneous mapping must be excluded.

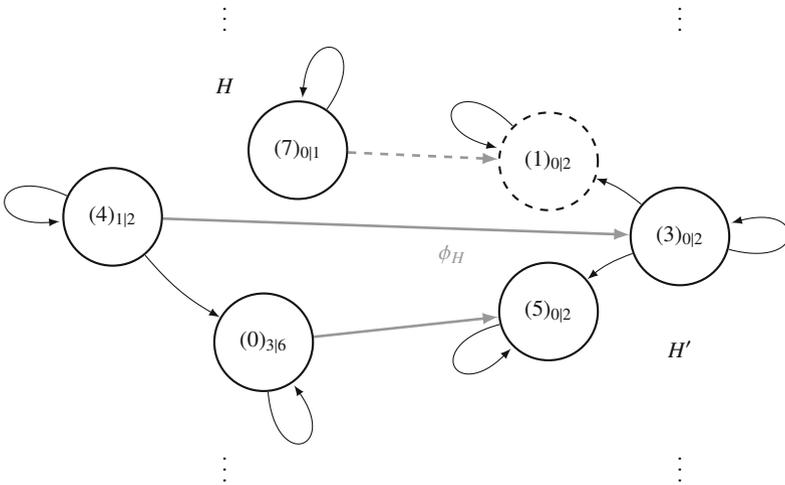


Fig. 5.4 There are enough vertices and arrows in H' to embed the two connected vertices $(4)_{1|2}$ and $(0)_{3|6}$ of H , but there is no disconnected vertex left to accept $(7)_{0|1}$.

Problem 5.1. *The more criteria applied to filter the mapping tree, the better the algorithm will perform during the next step. Can you think of more conditions based on (reasonably) simply computable digraph parameters?*

5.2.3.4 No Weak Mappings Without Children

A mapping that does not have children will be useless. For any weak mapping (w, w') , the existence of a compatible weak morphism $\phi_W : V_W \rightarrow V_{W'}$ such that $w' = \phi_W(w)$ implies the existence of a child morphism $\phi_H : V_H \rightarrow V_{H'}$. Because the component w is not empty, the set of successors in T_ϕ cannot be empty.

$$\forall h \subseteq w, \quad \exists (h, \phi_H(h)) \in N_{T_\phi}^+((w, w')). \quad (5.11)$$

Any weak mapping that does not have a child must be removed. The same holds for the full mapping, which must contain at least one weak mapping.

5.2.3.5 No Weak Mappings Without Local Combinations

A weak mapping is not valid simply because it has child mappings. It must also be possible to combine these homogeneous mappings to form a morphism locally.

Let (w, w') be a weak mapping. A combination of homogeneous mappings $C := \{(h_j, h'_j)\}_{j \in J}$, where $h_j \subseteq w$ and $h'_j \subseteq w'$ for any $j \in J$, forms a valid local combination if there exists a morphism restriction $\phi_H|_w : w \rightarrow w'$ such that for all $j \in J, h'_j = \phi_H|_w(h_j)$. Such a morphism exists if the collection C meets locally the same three sufficient *requirements* as defined for an entire homogeneous morphism in Section 4.3.3, namely:

1. The function $\phi_H|_w$ has to be well-defined. Every homogeneous component needs an image:

$$\forall h \in w, \quad \exists j \in J \text{ such that } h = h_j \text{ and } (h_j, h'_j) \in C. \quad (5.12)$$

2. The function $\phi_H|_w$ has to be injective:

$$\forall j_0, j_1 \in J, \quad j_0 \neq j_1 \Rightarrow h'_{j_0} \neq h'_{j_1}. \quad (5.13)$$

3. The function $\phi_H|_w$ has to be structure-preserving:

$$\forall j_0, j_1 \in J, \quad (h_{j_0}, h_{j_1}) \in A_H \Leftrightarrow (h'_{j_0}, h'_{j_1}) \in A_{H'}. \quad (5.14)$$

The weak morphism need not be injective. But it must be well-defined and non-adjacency-preserving (all weak components are disconnected). This means that the full mapping must have enough child mappings to cover all source components. In addition to single weak mappings, sets of mappings sharing a common target $\{(w_i, w')\}_{i \in I}$ must be considered too. Assume they allow simultaneous combinations, and then there should be a morphism $\phi_H : V_H \rightarrow V_{H'}$ such that for any $i \in I, \forall h \subseteq w_i$,

$$(h, \phi_H(h)) \in N_{T_\phi}^+((w_i, w')). \quad (5.15)$$

5.3 Procedure

Having stated the general principles underlying strict morphism enumeration, we now describe its implementation. The algorithm uses the component tree T_Q to model its input, relies internally on the mapping tree T_ϕ to keep track of valid mappings, and outputs an associative map M_ϕ giving the list of local combinations included in any set of quotient mappings. All data structures make extensive use of quotient digraphs and their problem-size reducing properties (see Section 5.2). The different steps of the procedure are detailed in chunks of pseudocode. We favored simplicity over efficiency, hoping the code is readable and shows the essential steps

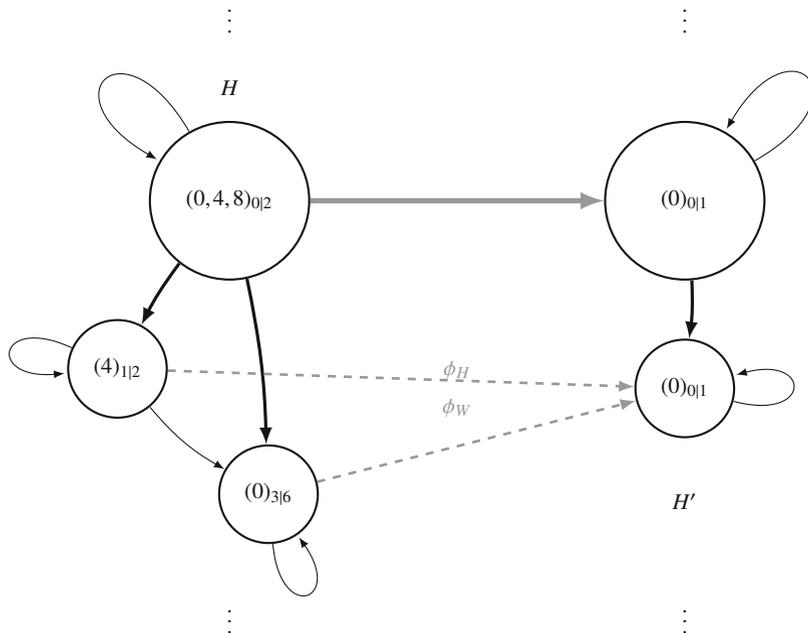


Fig. 5.5 Weak mappings involve singletons, so their mappings are not governed by many restrictions. Mapping the weak component $(0, 4, 8)_{0|2}$ to $(0)_{0|1}$ causes a problem at the homogeneous level. The two child components $(4)_{1|2}$ and $(0)_{3|6}$ cannot simultaneously be mapped to $(0)_{0|1}$ while preserving the structure of H . Since all child components must be mapped in order to build a local combination as stated by (5.12), this weak mapping does not possess any local combination and is therefore invalid.

of the procedure. An optimized implementation is left to the reader as an exercise. The figures of this section illustrate the algorithm’s action through the successive transformations of the example counterpoint of [Figure 5.6](#).

At the beginning, the local combinations map is empty.

$$M_\emptyset \leftarrow \emptyset;$$

Every attempt to retrieve a set of local combinations would yield an empty set. If a morphism exists, it is the task of the algorithm to populate this associative map with local combinations for each single weak mapping or group of non-injective weak mappings.

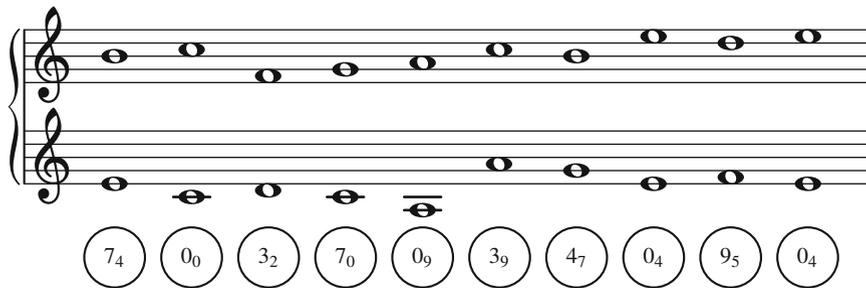


Fig. 5.6 A Phrygian counterpoint S taken from [83]: example number 67a, page 12. The cantus firmus is on the lower staff, the discantus on the upper staff. The corresponding contrapuntal consonances are displayed at the bottom, calculated with C as the reference pitch class. The following figures show how this piece of music gets encoded and transformed during the morphing process.

5.3.1 Constructing the Strict Digraphs

Application of Definition 4.4 yields the two strict digraphs D and D' associated with the source and target counterpoint worlds $\mathcal{W} = (\kappa, \sigma, p_A^\bullet)$ and $\mathcal{W}' = (\kappa', \sigma', p_{A'}^\bullet)$. To build the digraph D associated with the source world \mathcal{W} , we collect vertices and arrows.

```

1 // Start with empty strict digraph.
2  $D \leftarrow \emptyset$ ;
3 // Add every consonance to vertex set.
4 foreach  $\xi$  in  $\kappa^{-1}(1)$  {
5      $V_D \leftarrow V_D \cup \{\xi\}$ ;
6 }
7 // Add every forbidden step between consonances
8 // to the arrow set.
9 foreach  $(\xi_0, \xi_1)$  in  $\sigma^{-1}(0) \cap \kappa^{-1}(1)$  {
10      $A_D \leftarrow A_D \cup \{(\xi_0, \xi_1)\}$ ;
11 }

```

The source digraph generated by the example counterpoint is shown in [Figure 5.7](#). The same construction must be applied to the target digraph D' .

5.3.2 Constructing the Quotient Digraphs

The algorithm uses three of the five quotient relations defined in Section 4.3.1. The strong relation is omitted due to its similarity to the homogeneous relation, and the null relation can be generated on demand. [Table 5.2](#) summarizes the notation used

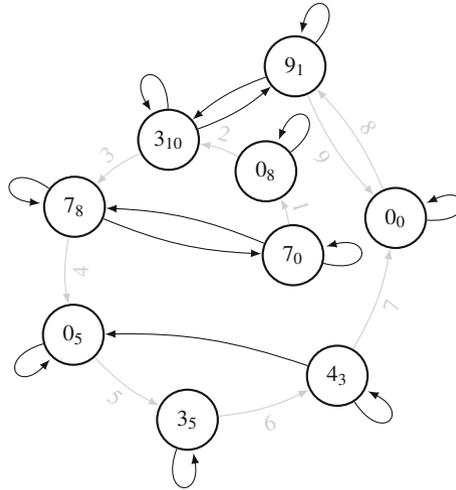


Fig. 5.7 Source strict digraph D of the local counterpoint world W related to counterpoint S of Figure 5.6. The digraph D is a subdigraph of the entire Fuxian digraph induced by the nine steps and intervals of the counterpoint, which forms a path in the complement digraph of D (see Section 4.2). Contrapuntal steps, drawn in gray, involve non-adjacent intervals. The digraph’s arrows (forbidden steps) are drawn in black.

throughout this chapter to designate the different quotient structures, as well as the ordering, which follows Section 4.3.2.6.

Table 5.2 Quotient level l and notations for source and target quotient digraphs $Q^{(l)}$ and $Q'^{(l)}$, related to the source and target strict digraphs, D respectively D' .

| l | Type | $Q^{(l)}$ | $Q'^{(l)}$ |
|-----|-------------|-----------|------------|
| 0 | Full | F | F' |
| 1 | Weak | W | W' |
| 2 | Homogeneous | H | H' |
| 3 | Null | N | N' |

We need the full, weak, homogeneous, and null quotient digraphs of the source and target worlds. The respective equivalence relations and constructions are explained in Section 4.3. The same procedure applies to each of the four levels.

```

1 // Process source and target digraphs,
2 foreach E in {D,D'} {
3     // with each equivalence relation.
4     foreach ~q in {~f, ~w, ~h, ~n} {

```

```

5      // Start with an empty quotient digraph.
6       $Q \leftarrow \emptyset$ ;
7      // Add every component to the vertex set.
8      foreach  $\xi$  in  $V_E$  {
9           $V_Q \leftarrow V_Q \cup \{E[[\xi]_{\sim_q}]\}$ ;
10     }
11     // Join every connected component.
12     foreach  $(\xi_0, \xi_1)$  in  $A_E$  {
13          $A_Q \leftarrow A_Q \cup \{E[[\xi_0]_{\sim_q}], E[[\xi_1]_{\sim_q}]\}$ ;
14     }
15 }
16 }

```

As soon as the two homogeneous digraphs are built, we verify whether a morphism can exist at all by applying condition (5.7).

```

1 // Does source homogeneous digraph fit
2 // into target ?
3 if  $|V_H| > |V_{H'}$  or  $|A_H| > |A_{H'}$  {
4     // If not, stop here.
5     return  $\emptyset$ ;
6 }

```

If the condition is not met, the algorithm halts and returns an empty list of local combinations. The Fuxian world's subgraph induced by the example counterpoint generates the homogeneous digraph shown in [Figure 5.9](#).

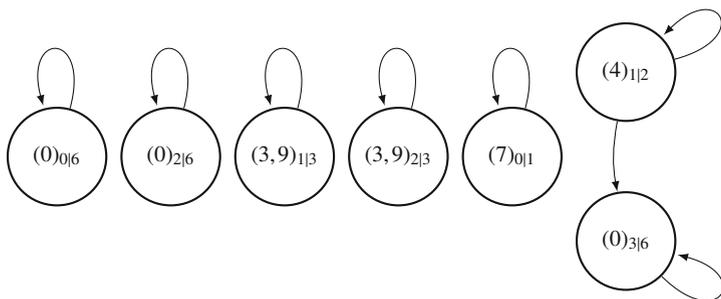


Fig. 5.8 Homogeneous digraph H of example counterpoint S . The labeling of components reflects the content of the global world, i.e., all intervals that could belong to the component, even if they are not traversed by S .

5.3.3 Constructing the Component Trees

Quotient components must be organized into a tree hierarchy, where parents are direct containers. We construct the source component tree T_Q by successively adding components from the broadest to the finest level.

```

1 // Start with empty component tree.
2  $T_Q \leftarrow \emptyset$ ;
3 // Scan through levels of inclusion hierarchy.
4 foreach  $l$  in  $\{0,1,2,3\}$  {
5     // Add every component of current level.
6     foreach  $q^{(l)}$  in  $V_{Q^{(l)}}$  {
7          $V_{T_Q} \leftarrow V_{T_Q} \cup \{q^{(l)}\}$ ;
8         // Link component to enclosing component.
9         if  $l > 0$  {
10            foreach  $q^{(l-1)}$  in  $V_{Q^{(l-1)}}$  {
11                if  $q^{(l)} \subseteq q^{(l-1)}$  {
12                     $A_{T_Q} \leftarrow A_{T_Q} \cup \{(q^{(l-1)}, q^{(l)})\}$ ;
13                    break;
14                }
15            }
16        }
17    }
18 }
```

The tree of the source world is shown in [Figure 5.9](#), the same procedure holds for the target world. The two structures model the algorithm's input.

5.3.4 Populating the Mapping Tree

To be useful, the mapping tree T_ϕ must contain only valid arrows. We must filter the mappings, i.e., keep the combinations of source and target components that belong to a morphism (see [Figure 5.10](#)). As explained in Section 5.2.3, this must be done in two steps. The first one consists of screening the combinatorial space to keep only those mappings that satisfy the necessary conditions. They rely on the comparison of simple digraph parameters described in Sections 5.2.3.1 to 5.2.3.3.

```

1 // Start with empty tree.
2  $V_{T_\phi} \leftarrow \emptyset$ ;
3 // Add root at full level.
4  $V_{T_\phi} \leftarrow V_{T_\phi} \cup \{(f, f')\}$ ;
5 // Add all combinations of weak components
6 // for which the target can hold the source.
```

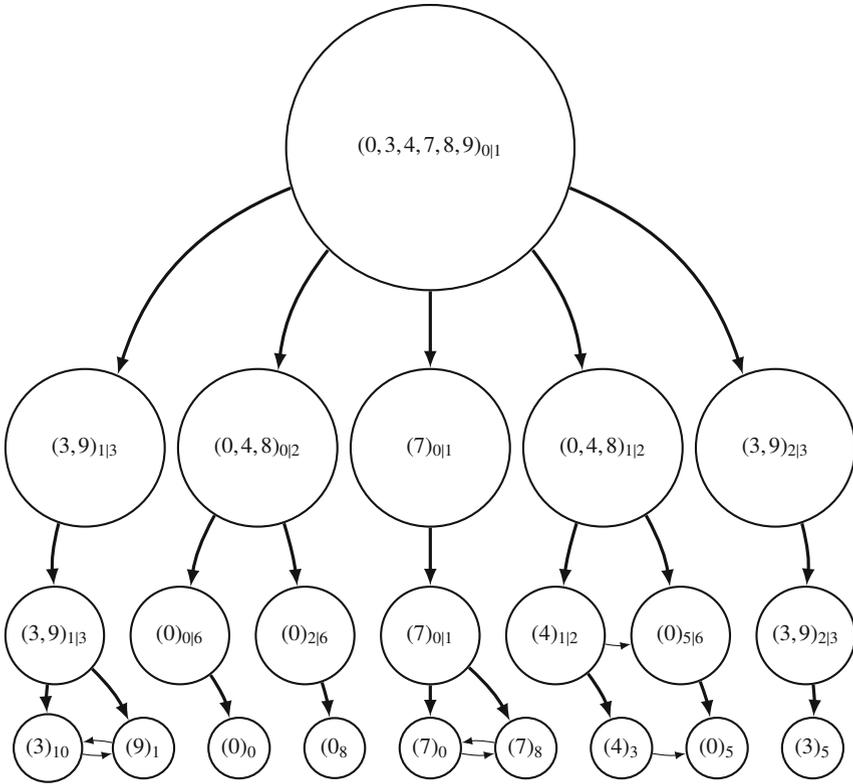


Fig. 5.9 Hierarchical organization of quotient components of the local Fuxian world into a quotient graph tree T_Q . Each node represents a quotient component, and thick arrows represent containment relations. Thin arrows are not part of the tree—they indicate connections in the quotient graphs. The tree reflects the quotient hierarchy: The root on top is the full component, its children the weak components, the grandchildren the homogeneous components, and the leaves at the bottom the null components, i.e. the contrapuntal intervals themselves.

```

7  foreach  $w \in V_W$  {
8      foreach  $w' \in V_{W'}$  {
9          // Verify weak mapping embedding condition (5.8).
10         if  $|V_{H[w]}| \leq |V_{H'[w']}|$  and  $|A_{H[w]}| \leq |A_{H'[w']}|$  {
11             // Add weak component.
12              $V_{T_\phi} \leftarrow V_{T_\phi} \cup \{(w, w')\}$ ;
13             // Join to full parent.
14              $A_{T_\phi} \leftarrow A_{T_\phi} \cup \{((f, f'), (w, w'))\}$ ;
15             // Add all combinations of child components
16             // satisfying conditions (5.9) and (5.10).
17             foreach  $h \subseteq w$  {
18                 foreach  $h' \subseteq w'$  {

```

```

19         if  $d_H^-(h) \leq d_{H'}^-(h')$  and
20            $d_H^+(h) \leq d_{H'}^+(h')$  and
21              $|V_{H[V_H \setminus N_H(h)]}| \leq |V_{H'[V_{H'} \setminus N_{H'}(h')]}|$  and
22                $|A_{H[V_H \setminus N_H(h)]}| \leq |A_{H'[V_{H'} \setminus N_{H'}(h')]}|$  {
23                 // Add homogeneous component.
24                  $V_{T_\phi} \leftarrow V_{T_\phi} \cup \{(h, h')\}$ ;
25                 // Join to weak parent.
26                  $A_{T_\phi} \leftarrow ((w, w'), (h, h'))$ ;
27             }
28         }
29     }
30 }
31 }
32 }

```

The preliminary filtering leaves us with an overpopulated tree: T_ϕ contains some quotient mappings that passed the first tests but do not belong to a quotient morphism. It is the purpose of this quick and imperfect filtering process to prepare T_ϕ for later cleaning. If most of the invalid mappings have already been filtered out, the algorithm should spend less time in the final cleaning step, which involves many more computations.

5.3.5 Populating the Combinations Map

As mentioned in Section 5.2.2, weak mappings do not provide as safe a guide for homogeneous mappings as homogeneous mappings do for strict mappings. It can happen that two or more weak components get mapped to the same target component. In parallel to the (internal) mapping tree T_ϕ , we maintain the output data structure, the map of local combinations M_ϕ .

Following the same strategy as for the mapping tree (overpopulate the tree and then filter it), we recursively add local combinations associated to the full and weak mappings.

```

1 // Start at full level and investigate
2 // combinations of weak mappings.
3 populate_locomb((f, f'));
4 // Proceed recursively along the branches of  $T_\phi$ .
5 function populate_locomb(( $q^{(l-1)}, q'^{(l-1)}$ )) {
6     // Find all groups of mappings pointing at  $q'^{(l-1)}$ .
7      $P \leftarrow$  collect_parents( $q'^{(l-1)}$ );
8     // Store local child combinations.
9     foreach  $\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}$  in  $P$  {
10         // Generate all local combinations

```

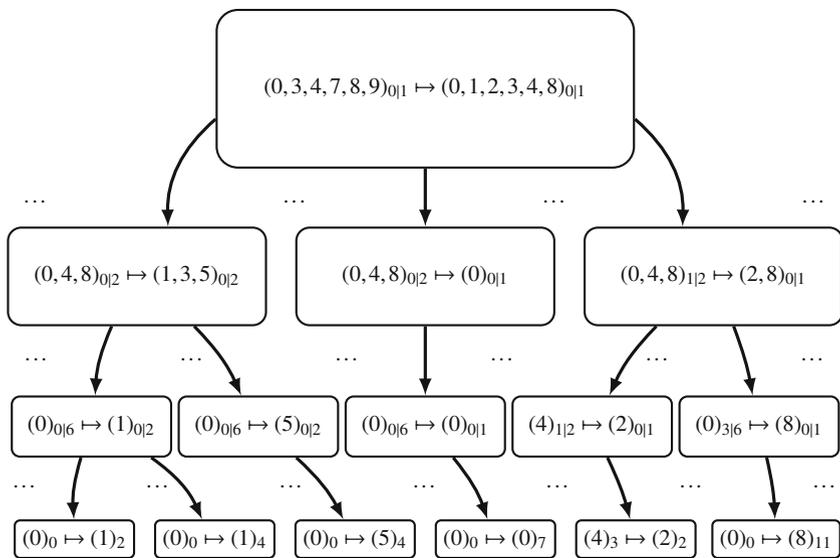


Fig. 5.10 Partial representation of the quotient mapping tree T_ϕ of the local world induced by the example counterpoint into the twelve-tone target world of class number 68 in [57]. The single full quotient mapping is located on top, null quotient mappings at the bottom. At each level, sets of children nodes contain all alternative mappings of a same source node. For example, the weak component $(0, 4, 8)_{0|2}$ can be mapped to $(1, 3, 5)_{0|2}$, $(0)_{0|1}$, or $(1, 3, 5)_{1|2}$ (the last one is not shown). On the other hand, the weak component $(0, 4, 8)_{1|2}$ can only be mapped to $(2, 8)_{0|1}$. Many null mappings are compatible with a given homogeneous mapping. Here, only two among the six possible null mappings $(0)_0 \mapsto (1)_{2x}$ are shown. Components are labeled after the global worlds, even if local worlds contain a small subset of vertices. In this example, the weak component $(0, 4, 8)_{0|2}$ contains only the homogeneous components $(0)_0$ and $(0)_8$. All numbers denote congruence classes modulo 12.

```

11         // and store them.
12          $M_\phi [ \{ \{ (q_i^{(l-1)}, q'^{(l-1)}) \}_{i \in I} ]$ 
13              $\leftarrow \text{create\_locomb}(\{ \{ (q_i^{(l-1)}, q'^{(l-1)}) \}_{i \in I} ) ;$ 
14     }
15     // Continue with every child mapping.
16     foreach  $(q^l, q'^l)$  in  $N_{T_\phi}^+(\{(q^l, q'^l)\})$  {
17         populate\_locomb $(\{(q^l, q'^l)\}) ;$ 
18     }
19     // Recursion stops at leaves.
20 }
```

5.3.5.1 Collecting Parent Combinations

When visiting a new node $(q^{(l-1)}, q'^{(l-1)})$, the first task of the map-populating procedure is to collect all combinations of mappings pointing simultaneously at the same target component $q^{(l-1)}$.

```

1 function collect_parents ( $q^{(l-1)}$ ) {
2     // Start with empty set of parent mappings.
3      $U \leftarrow \emptyset$ ;
4     // Inspect each mapping at same level.
5     foreach ( $q, q'$ ) in  $V_{T_\phi}^{(l-1)}$  {
6         // If it is pointing at  $q^{(l-1)}$ ,
7         if  $q' = q^{(l-1)}$  {
8             // add it.
9              $U \leftarrow U \cup \{(q, q')\}$ ;
10        }
11    }
12    // Possible groups of parent mappings
13    // are subsets of the set of all parent mappings.
14     $P \leftarrow \emptyset$ ;
15    foreach  $\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}$  in  $2^U$  {
16        // Add the new subset.
17         $P \leftarrow P \cup \{\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}\}$ ;
18    }
19    // Return the set of subsets.
20    return  $P$ ;
21 }
```

5.3.5.2 Computing Child Combinations

Once we have a set of parent mappings $\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}$, we generate the included combinations of child mappings $\Phi_{Q^{(l)}} |_{\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}}$.

```

1 function create_locomb ( $\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}$ ) {
2     // Start with an empty set of combinations.
3      $\Phi_{Q^{(l)}} |_{\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}} \leftarrow \emptyset$ ;
4     // Collect all source components.
5      $\{q_j^{(l)}\}_{j \in J} \leftarrow N_{T_Q}^+(\cup_{i \in I} q_i^{(l-1)})$ ;
6     // Inspect every source component.
7     foreach  $j$  in  $J$  {
8         // Collect all mappings  $(q_j^{(l)}, q'_{k_j}^{(l)})$ 
9         // available to source component  $q_j^{(l)}$ .
```

```

10       $\{(q_j^{(l)}, q_{k_j}^{\prime(l)})\}_{k_j \in K_j} \leftarrow \{(q, q') \in N_{T_\phi}^+ (\{(q_i^{(l-1)}, q_i^{\prime(l-1)})\}_{i \in I}) : q = q_j^{(l)}\}$ ;
11      // Verify that there is
12      // at least one mapping available.
13      if  $K_j = \emptyset$  {
14          // If not, there is no well-defined
15          // combination (5.12).
16          return  $\emptyset$ ;
17      }
18  }
19  // Try to assemble every possible combination.
20  foreach  $(k_j)_{j \in J}$  in  $\times_{j \in J} K_j$  {
21      // Verify its validity according to
22      // criteria of level  $l$ .
23      is_valid_combination  $\leftarrow$  false;
24      if  $l = 1$  {
25          // Weak combination.
26          is_valid_combination
27               $\leftarrow$  check_weak_comb  $((q_j^{(l)}, q_{k_j}^{\prime(l)})_{k_j \in K_j})$ ;
28      } else if  $l = 2$  {
29          // Homogeneous combination.
30          is_valid_combination
31               $\leftarrow$  check_homog_comb  $((q_j^{(l)}, q_{k_j}^{\prime(l)})_{k_j \in K_j})$ ;
32      } else {
33          // Do not handle other levels.
34      }
35      // Store combination if it is valid.
36      if is_valid_combination  $=$  true {
37           $\Phi_{Q^{(l)}|_{(q^{(l-1)}, q^{\prime(l-1)})}} \leftarrow \Phi_{Q^{(l)}|_{(q^{(l-1)}, q^{\prime(l-1)})}} \cup \{(q_j^{(l)}, q_{k_j}^{\prime(l)})_{k_j \in K_j}\}$ ;
38      }
39  }
40  // Return all stored valid combinations.
41  return  $\Phi_{Q^{(l)}|_{(q^{(l-1)}, q^{\prime(l-1)})}}$ ;
42 }

```

Problem 5.2. Implement the combination enumeration step $(k_j)_{j \in J}$ in $\times_{j \in J} K_j$.

Problem 5.3. Generating local combinations implies trying all combinations of mappings, a time-consuming procedure. A basic rule of optimization is to avoid doing the same computation twice. How can you organize your data so that the code does not repeatedly compute combinations included in a single weak mapping (w, w') appearing in several parent combinations?

5.3.5.3 Filtering Weak Combinations

Weak morphisms must be well-defined. There are no other simple conditions that can be computed at this level, besides looking at valid child combinations at the homogeneous level. Nevertheless, some preliminary filtering can be achieved by checking that non-injective combinations point at targets that can hold all sources.

```

1  function check_weak_comb  $((w_j, w'_j)_{j \in J})$  {
2      // Assume combination is valid.
3      combination_is_valid  $\leftarrow$  true;
4      foreach  $j$  in  $J$  {
5          // Count source components and arrows.
6           $v \leftarrow 0$ ;
7           $a \leftarrow 0$ ;
8          // Sum over number of vertices and arrows
9          // to embed into target component.
10         foreach  $k$  in  $J$  {
11             if  $w'_k = w'_j$  {
12                  $v \leftarrow v + |V_{H[w_k]}|$ ;
13                  $a \leftarrow a + |A_{H[w_k]}|$ ;
14             }
15         }
16         // Count target components and arrows.
17          $v' \leftarrow |V_{H'[w'_j]}|$ ;
18          $a' \leftarrow |A_{H'[w'_j]}|$ ;
19         // If target component cannot hold simultaneously
20         // all source components, combination is invalid.
21         if  $G \geq a'$  or  $v \geq v'$  {
22             combination_is_valid  $\leftarrow$  false;
23         }
24     }
25     return combination_is_valid;
26 }

```

5.3.5.4 Filtering Homogeneous Combinations

In addition to being well-defined, a homogeneous morphism must be injective (5.13) and structure-preserving (5.14).

```

1  function check_homog_comb  $((h_j, h'_j)_{j \in J})$  {
2      // Assume combination is valid.
3      combination_is_valid  $\leftarrow$  true;
4      foreach  $j$  in  $J$  {

```

```

5     foreach k in J \ {j} {
6         // Is mapping pair injective?
7         if h'_k = h'_j {
8             combination_is_valid ← false;
9         }
10        // Is mapping pair structure-preserving?
11        if ((h_k, h_j) ∈ A_H
12            and (h'_k, h'_j) ∉ A_{H'})
13            or ((h_k, h_j) ∉ A_H
14                and (h'_k, h'_j) ∈ A_{H'}) {
15            combination_is_valid ← false;
16        }
17    }
18 }
19 return combination_is_valid;
20 }

```

5.3.6 Filtering the Quotient Structures

At this stage, the tree T_ϕ may have been filled with mappings and the map M_ϕ with local combinations. We must finish the filtering work using sufficient conditions to decide whether each mapping and combination is valid.

5.3.6.1 Iteration Verifications

If a mapping is detected as being invalid, it needs to be removed from T_ϕ and M_ϕ . This affects entire branches of T_ϕ and combinations in M_ϕ , whose validity need to be checked again. The procedure follows alternating removal and verification steps. During each pass, the partially filtered T_ϕ is scanned from the full level down to the homogeneous level. If a mapping fails the validity test, it is removed, local combinations in M_ϕ are updated accordingly, and the next pass is triggered. Intercepting invalid mappings at the weak level allows one to clean the tree more efficiently by getting rid of all contained homogeneous mappings at once, but most of the invalid mappings can only be spotted at the homogeneous level, which requires generating combinations.

Iteration stops after a finite number of passes. At the beginning, T_ϕ contains a possibly huge but finite number of mappings. Each pass removes zero, one, or several mappings. The algorithm stops if the tree has been completely emptied (there are no valid mappings left, so no valid morphisms) or if no invalid mappings have been found (no change during the last pass means no mapping removed, i.e., T_ϕ

contains only valid mappings). Mappings are inspected for validity according to (locally) sufficient criteria.

```

1 // Vertical filtering loop: multiple passes.
2 // Continue as long as the tree is not empty.
3 while  $T_\phi \neq \emptyset$  {
4     // Save previous state of  $T_\phi$ .
5      $T_\phi^{old} \leftarrow T_\phi$ ;
6     // Verify that the tree does not contain
7     // invalid mappings.
8     inspect_mappings( $(f, f')$ );
9     // Has the tree changed?
10    if ( $T_\phi = T_\phi^{old}$ ) {
11        // Final state has been reached.
12        break;
13    }
14 }
```

5.3.6.2 Detecting Invalid Mappings

Two conditions can be used to detect *false positives* that passed the preliminary validity checks of Section 5.3.4. Both involve parent and child quotient mappings, located at the weak and homogeneous levels. The conditions each mapping must satisfy to stay in T_ϕ are described in Sections 5.2.3.4 and 5.2.3.5. This is the loop's verification stage. It triggers the removal of any invalid mapping it detects and further verifications of the remaining structures.

```

1 // One pass: follow recursively branches
2 // of mapping tree, and filter vertically.
3 function inspect_mappings( $(q^{(l)}, q'^{(l)})$ ) {
4     // Assume mapping is not valid.
5     is_mapping_valid ← false;
6     // Verify existence of children (5.11).
7     if  $N_{T_\phi}^+((q^{(l)}, q'^{(l)})) \neq \emptyset$  {
8         // Verify existence of valid local combinations
9         (5.15).
10        // Is there at least one set of mappings
11        // containing  $(q^{(l)}, q'^{(l)})$  and
12        // having valid child combinations?
13         $P \leftarrow \text{collect\_parents}(q'^{(l)})$ ;
14        foreach  $\{(q_i, q'_i)\}_{i \in I}$  in  $P$  {
15            if  $M_\phi[\{(q_i, q'_i)\}_{i \in I}] \neq \emptyset$  {
16                is_mapping_valid ← true;
17                break;
18            }
19        }
20    }
21 }
```

```

17         }
18     }
19 }
20 // Is mapping valid?
21 if mapping_is_valid=false {
22     // Remove invalid mapping.
23     remove_mapping(( $q^{(l)}$ ,  $q'^{(l)}$ ));
24 }
25 // Inspect children.
26 // Recursion will stop at leaves.
27 foreach ( $q^{(l+1)}$ ,  $q'^{(l+1)}$ )  $\in N_{T_\phi}^+$ (( $q^{(l)}$ ,  $q'^{(l)}$ )) {
28     inspect_mappings(( $q^{(l+1)}$ ,  $q'^{(l+1)}$ ));
29 }
30 }

```

5.3.6.3 Removing Invalid Mappings and Combinations

When a mapping is removed, all related structures need to be updated. This implies child mappings and local combinations.

```

1 // Clean up  $T_\phi$  and  $M_\phi$ .
2 function remove_mapping(( $q^{(l)}$ ,  $q'^{(l)}$ )) {
3     // Remove all child mappings.
4     foreach ( $q^{(l+1)}$ ,  $q'^{(l+1)}$ )  $\in N_{T_\phi}^+$ (( $q^{(l)}$ ,  $q'^{(l)}$ )) {
5         remove_mapping(( $q^{(l+1)}$ ,  $q'^{(l+1)}$ ));
6     }
7     // Remove all related combinations.
8     // Inspect every group of multiple combinations.
9      $P \leftarrow \text{collect\_parents}(q'^{(l-1)})$ ;
10    foreach  $\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}$  in  $P$  {
11         $\Phi_{Q^{(l)}}|_{\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}} \leftarrow M_\phi[\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}]$ 
12        // Inspect every local combination.
13        foreach  $\phi_{Q^{(l)}}|_{\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}}$  in  $\Phi_{Q^{(l)}}|_{\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}}$  {
14            // Combination contains invalid mapping,
15            if ( $q^{(l)}$ ,  $q'^{(l)}$ )  $\in \phi_{Q^{(l)}}|_{\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}}$  {
16                // Remove combination.
17                 $\Phi_{Q^{(l)}}|_{\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}} \leftarrow \Phi_{Q^{(l)}}|_{\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}}$ 
18                     $\setminus \{\phi_{Q^{(l)}}|_{\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}}\}$ ;
19            }
20        }
21    }

```

```

22 // Remove node (and incident arrow) from  $T_\phi$ .
23  $V_{T_\phi} \leftarrow V_{T_\phi} \setminus \{(q^{(l)}, q'^{(l)})\};$ 
24 }

```

Parent mappings $(q^{(l-1)}, q'^{(l-1)})$ may lose their only child mapping $(q^{(l)}, q'^{(l)})$. Sets of local combinations $\Phi_{Q^{(l)}}|_{\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}}$ may be emptied by this procedure. If a parent node becomes invalid, it will be detected during the next iteration and removed as well. A removal cascade can propagate up to the root of T_ϕ , indicating that no morphism exists.

5.3.7 Listing Valid Mappings

The algorithm outputs not ready-to-use morphisms but building blocks for morphisms. They guide the construction of morphisms by successive addition of mappings, as described in Section 5.1. From a user's point of view, what is needed at each step l is help for the next step. An entire set of partial combinations may be overwhelming, but a simple list of the next valid candidate mappings may help: The user chooses a source component, h_l and the computer gives a list of all mappings that can be safely added to the existing set $\phi_H^{(l-1)}$, as shown in [Figure 5.11](#).

```

1 function list_mappings( $\phi_H^{(l-1)}$ ,  $h_l$ ) {
2 // Start with an empty set of mappings.
3  $K \leftarrow \emptyset;$ 
4 // Inspect every local combination.
5 foreach  $\Phi_H|_{\{(w_i, w')\}_{i \in I}}$  in  $M_\Phi$  {
6 foreach  $\phi_H|_{\{(w_i, w')\}_{i \in I}}$  in  $\Phi_H|_{\{(w_i, w')\}_{i \in I}}$  {
7 // If it contains the existing combination,
8 if  $\phi_H^{(l-1)} \subset \phi_H|_{\{(w_i, w')\}_{i \in I}}$  {
9 // Add target to list.
10  $K \leftarrow K \cup \{\phi_{Q^{(l)}}|_{\{(q_i^{(l-1)}, q'^{(l-1)})\}_{i \in I}} | h_l\};$ 
11 }
12 }
13 }
14 return  $K;$ 
15 }

```

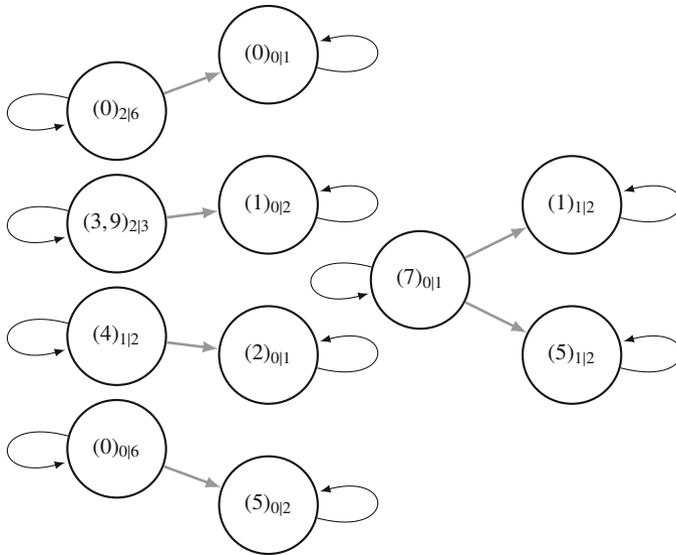


Fig. 5.11 Given a set of mappings $\phi_H^{(4)} = \{((0)_{2|6}, (0)_{0|1}), ((3, 9)_{2|3}, (1)_{0|2}), ((4)_{1|2}, (2)_{0|1}), ((0)_{0|6}, (5)_{0|2})\}$ already assigned to the construction of ϕ_H , possible mappings from $(7)_{0|1}$ are $\{((7)_{0|1}, (1)_{1|2}), ((7)_{0|1}, (5)_{1|2})\}$.

5.3.8 Generating Strict Mappings

As mentioned in Section 5.2.1, given a homogeneous mapping, it is straightforward to generate the included strict mappings. Due to the huge number of strict mappings and the lack of restrictions governing their combinations, calculating instead of storing local strict combinations saves a lot of space, see [Figure 5.12](#).

```

1 function generate_strict_comb ((h,h')) {
2     // Start with an empty set.
3      $\phi|_{(h,h')} \leftarrow \emptyset$ ;
4     // Add mapping from every source vertex,
5     foreach  $\xi$  in h {
6         // to every target vertex.
7         foreach  $\xi'$  in h' {
8              $\phi|_{(h,h')} \leftarrow \phi|_{(h,h')} \cup \{(\xi, \xi')\}$ ;
9         }
10    }
11    return  $\phi|_{(h,h')}$ ;
12 }
```

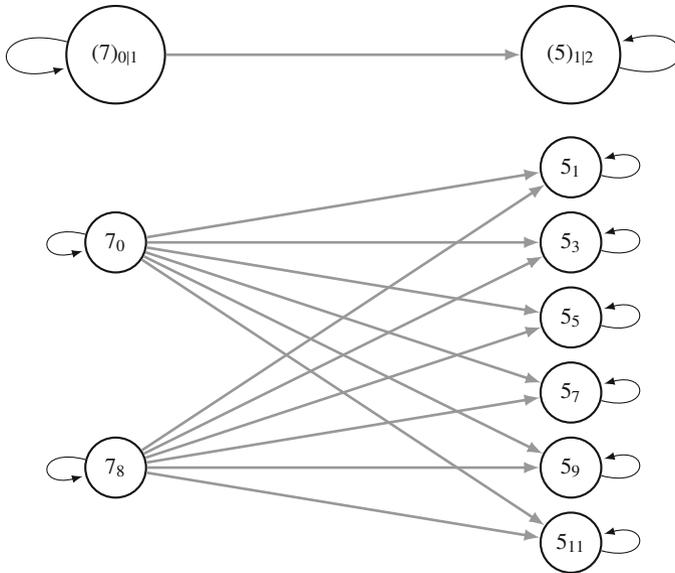


Fig. 5.12 Local strict combinations for the homogeneous mapping $((7)_{0|1}, (5)_{1|2})$. The two source consonances 7_0 and 7_8 can be freely mapped to any of the six target consonances 5_{2x+1} . Strict local combinations belong to the list of 36 pairs of mappings $\{(7_0, 5_1), (7_8, 5_1)\}, \{(7_0, 5_3), (7_8, 5_1)\}, \dots, \{(7_0, 5_{11}), (7_8, 5_9)\}, \{(7_0, 5_{11}), (7_8, 5_{11})\}$.

5.4 Discussion

We proposed a way of listing and organizing graph matchings that can be used for computer-assisted composition. Using the map of local combinations M_ϕ , one can safely build a transformation by assembling assignments of intervals from one world to another. The resulting construction will always yield a complete valid morphism. Whatever the assignments chosen from this list, it is always possible to map every interval without breaking the counterpoint rules in the target world. Section 6.2.3 illustrates how the role of the computer is to filter out invalid choices and dead ends, while the role of the user is to choose among the allowed possibilities.

5.4.1 Complexity

Worst-case execution times show an exponential dependence on the number of vertices. A more-precise estimate of the time needed by the algorithm to enumerate morphisms must take into account pruning, whose primary goal is to keep us as far away from the worst case as possible. A simple approximation of the number of combinations to test is given by

$$\prod_{h \in V_H} |\{h' \in V_{H'} : d_H^-(h) \leq d_{H'}^-(h') \text{ and } d_H^+(h) \leq d_{H'}^+(h')\}|, \quad (5.16)$$

which can serve as an easily computable proxy for measuring complexity. [Figure 5.13](#) shows different execution times compared with this gross estimate for different combinations of counterpoint worlds.

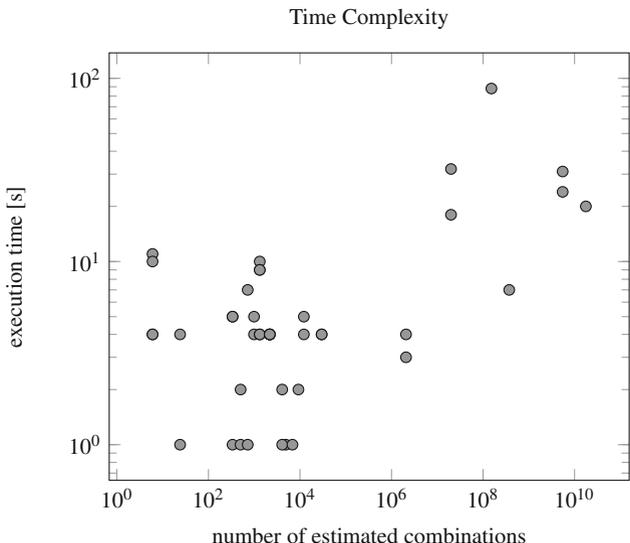


Fig. 5.13 Time required to enumerate the combinations of mappings estimated by formula (5.16) between different pairs of counterpoint worlds. Values were measured using the software of Chapter 6 and may vary from one machine to another. Only filtering criteria (5.9) is considered in the combinations estimate. By applying the remaining criteria, the algorithm may finish earlier, especially if it detects that no morphism can be constructed, hence the different execution times for a same number of combinations.

Another possible cause of increasing execution times is the number of target vertices. Because we are searching for subgraph isomorphisms and not just graph isomorphisms, the source vertices are not influential: the bigger the target digraph, the more possibilities to embed the source digraph. The user should be prepared also for longer enumerations in microtonal worlds. Some of them may generate bigger digraphs.

Internal symmetries, such as those found in the Fuxian world with its two 6-stars (see [Figure 4.16](#)), may increase the number of candidates for mappings. This too can lengthen the enumeration process.

5.4.2 Global Morphisms

We mentioned in the introduction to this chapter that the existence of global morphisms, i.e., transformations of an entire counterpoint world into another should not be taken for granted. While smaller, local worlds can often be transformed, global morphisms are an exception, and their existence depends on the structure of the worlds involved. Figure 5.14 shows how the six twelve-tone worlds relate to one another, and how the Fuxian world is completely isolated from the others, having its own distinguished contrapuntal structure.

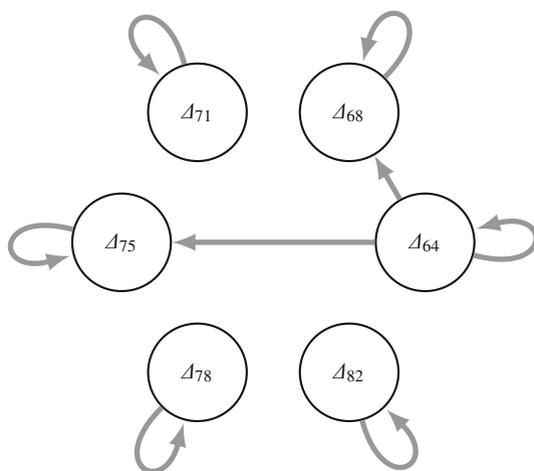


Fig. 5.14 Existence of global morphisms between the twelve-tone worlds is rather an exception. Automorphisms always exist, but among the 30 possible pairs of different worlds, only two yield a morphism. The Fuxian world Δ_{82} appears to be completely isolated from the other worlds: There is no global way in or out.

Nevertheless, a counterpoint world can also be similar to other worlds from a different octave division. For example, the twelve-tone world of class index 71 is isomorphic to eight of the 16-tone worlds (see Appendix B).

Chapter 6

Experimentation

The explicit expression of counterpoint rules has always attracted scientists and musicians interested in computer science. It does not come as a surprise that the first musical piece composed with the help of a computer was a counterpoint simulation.¹ The algorithmic flavor of the *Gradus* and its systematic organization into species [29] calls logically for a translation of composition techniques into formal mathematical models and algorithms. Many different software tools have been developed since the 1950s for fully automatic or computer-assisted counterpoint composition, covering almost the entire range of techniques used in artificial intelligence.

The earliest attempts all include rule-based systems such as Ebcioğlu's *Computer Counterpoint* [24] or Schottstaedt's *Automatic Species Counterpoint* [79, 81]. *PW-Constraints* is a more recent example of constraint programming [8]. The software presented in this chapter also fits into this rule-based paradigm.

During the last fifteen years, new techniques emerged. *GPMuse* uses a genetic algorithm [75], and more recently [1]; Cope's *Gradus* implements machine learning [18]; reinforcement learning is found in [74]; Farbood's *Palestrina* is based on Markov models [27]; neural networks are studied in [2]; combinatorial optimization is performed by *FuX* [36]; and even fuzzy logic is suggested in [95].

As summarized in [68], automatic generation of music has mainly two functions: simulating a style in order to validate musicological analyses and models, or acting as a tool for composing genuinely new material. The strength of our mathematical model is not so much to simulate Fuxian rules, but to enable the exploration of new musical territories based on a solid structural framework. The software tools presented in this chapter provide a way to investigate one of the main questions raised in this book: Successful use of counterpoint spans several centuries, but is it possible today to capitalize on this musical structure to create new material successfully?

It is hard to get an idea of what such musical worlds sound like without playing, trying, and hearing. All this is made possible through the tools whose manipulation is described in this chapter. Section 6.1 explains how to install the necessary soft-

¹ Lejaren Hiller's *Illiad suite* [40].

ware, Section 6.2 shows how to explore new counterpoint worlds, and Section 6.3 contains a more in-depth description of the software components.

6.1 Rubato

The counterpoint software implementing the theories developed in this book is platform-independent and should run on any computer. It is part of a broader ecosystem called *Rubato*, which is a Java-based host application that provides a working environment for mathematical and musical experimentation. The application's features can be expanded through a series of additional modules, each dedicated to a particular task, such as reading a MIDI file or performing transformations on note parameters. Increasingly complex problems can be solved by further combining these basic units. The modules dealing with counterpoint are only a part of the system.

6.1.1 Installation

Three different pieces of software need to be installed on the computer before counterpoint manipulations can be done: a system, a host application, and counterpoint-specific modules.

6.1.1.1 Java Virtual Machine

At the lowest level, a Java Runtime Environment (JRE) version 1.5 standard edition (SE) or higher provides the virtual machine and system libraries. Latest distributions for the most common operating systems can be retrieved from the official website:

<http://www.java.com/>

6.1.1.2 RUBATO Platform

At the intermediate level, a host application provides the mathematical framework and user interface. Download the latest binaries of the RUBATO Composer software stored at

<http://www.rubato.org/>

and uncompress the archive by double-clicking on the downloaded file or uncompress and extract the archive manually, using the shell commands.

```
$ gzip -d rubato-bin-20070720.tar.gz
$ tar -xf rubato-bin-20070720.tar
```

Then launch the Java application.

```
$ cd rubato-bin-20070720
$ java -jar rubato.jar
```

6.1.1.3 BollyFux Plug-ins

The counterpoint software is not part of the standard RUBATO distribution but is encapsulated in a suite of plug-ins that must be downloaded separately from the same website as the main application. RUBATO expects these additional modules to be stored inside a hidden directory located in your home directory, as described in [65]. The same procedure holds for any other module added to RUBATO. If this directory has not been created yet, it is the first thing to do before using this kind of add-ons.

```
    # Go to your home directory
$ cd
    # Create the hidden folders
$ mkdir ~/.rubato/plugins
    # Place the rubettes at the right place
$ mv ~/Downloads/BollyFux.jar ~/.rubato/plugins/
```

You now should have a working installation of RUBATO and be able to open the example documents from within the application.²

6.1.2 Quick Start

Every situation necessitates a specific setup of processing units. The user builds a machine for a particular task by choosing modules and combining them into a network to process musical data.

6.1.2.1 How to Choose a Rubette

Such a machine is assembled with several specialized pieces of software, usually called *plug-ins* or *modules*, *rubettes* in the RUBATO jargon. Some modules will perform exactly the task the user is looking for and be readily usable, while more sophisticated problems may require the combination of several modules. The catalog of available rubettes appears on the right side of the window, see [Figure 6.1](#).

² The RUBATO application is stored in a Java archive, and is therefore considered by some operating systems as a document (of the Java program itself). RUBATO document files (whose extension is .rbo) can be opened only from within the application, not by double-clicking its icon.

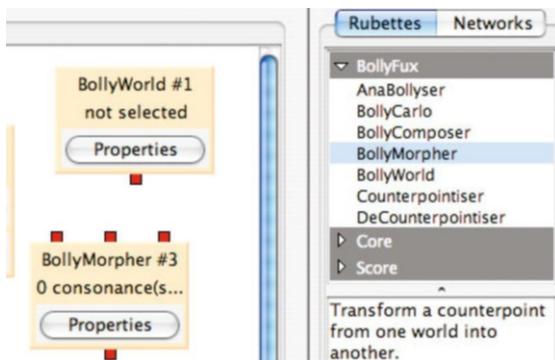


Fig. 6.1 The catalog of available rubettes appears on the right panel of the RUBATO window. The user chooses an item and adds it to the current network by dragging and dropping it onto the main pane or by double-clicking its name in the list. A small description of each module appears below when it is selected. Related modules are hierarchically organized in thematic collections whose content can be expanded or hidden. All counterpoint rubettes appear under the *BollyFux* entry. On the central network pane, an added module appears as a yellow box. A short description of the rubette’s role appears when the mouse hovers over the rubette’s area.

6.1.2.2 How to Connect Rubettes

Rubettes are only building blocks. They become useful once integrated into a larger system where they are assigned one task in a chain of operations. They need to be connected and form a processing graph where data can be exchanged and further transformed.

Each rubette has one or more input and output connectors. They allow the user to decide how to feed the rubette with data to process and what to do with the resulting data. The sequence of operations is defined by linking the processing units, as shown in [Figure 6.2](#).

6.1.2.3 How to Specify Parameters in a Rubette

Data exchanged between rubettes is computed on the fly. Besides this *dynamic* processing, there are also *static* parameters with fixed values that determine the general behavior of individual rubettes—as for instance MIDI files to load, general tempo settings, tuning, etc. All these values need to be set separately by the user. When a rubette needs a specific setting, it shows a *Properties* button inside its box. Clicking on it opens a dialog box where all the parameters can be set. There are always the same four buttons at the bottom of the dialog box.

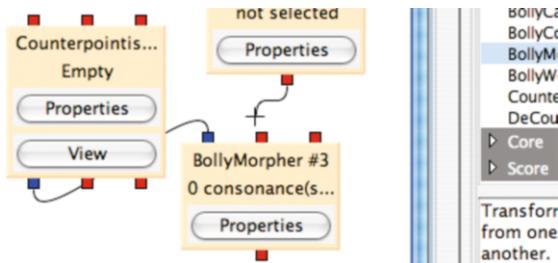


Fig. 6.2 In the central network pane, connectors appear as small boxes next to the rubettes. Inputs lie on top, outputs on the bottom. Connectors already used are displayed in blue, unused ones in red. It is possible to connect one output to multiple inputs (making *fan* connections). The user establishes a connection from one rubette to another by dragging an output to the input of the next rubette. The target connector is identified as soon as a black frame surrounds the rubette. Releasing the mouse then commits the connection. In the case of several connectors, it may be difficult to remember which one to choose. A small description of the type of data moving through the connector appears when the mouse hovers over it.

- Hide Closes the dialog box without changing any saved parameter. The values entered remain in the dialog’s fields without affecting the rubette’s operation.
- Revert Cancels any changes made by the user and replaces the dialog’s values with the ones saved in the rubette.
- Apply Commit the changes. The parameters used by the rubette now correspond to the values entered by the user. The dialog remains open for further parametrization.
- Apply & Hide Same as above, but also closes the dialog box.

6.1.2.4 How to Run a Network

Once the rubettes are properly connected and all parameters set, the actual processing can start. Nothing will happen until the user explicitly asks RUBATO to start running. The data will then be sent through the network, starting with source modules, like those loading MIDI files, all the way down to sink modules, such as those saving the transformed data into an other MIDI file or displaying the results on the screen. The software performs a single pass through the network and stops. The user is then free to inspect the computation’s results, either through RUBATO’s display dialogs, by pressing the View button of a rubette that allows data visualization, or by using external software—to inspect MIDI or Csound files, for example.

6.1.2.5 Saving and Retrieving RUBATO Machines

A network configuration (modules and connections) can be saved along with their parameters into a RUBATO document file, identified by the `.rbo` extension. The processed data is not part of the document, so the network needs to be run again after the document has been loaded. Simply use the usual Open and Save commands of the File menu.

6.2 Recipes

Four different cases are suggested for becoming familiar with the counterpoint software. They cover different usages: automatic generation in Section 6.2.1, computer-assisted composition in Section 6.2.2, and transformation in Section 6.2.3, which is the topic of Chapter 5. The last case in Section 6.2.4 suggests how these tools may be integrated into a broader environment. Ready-to-use implementations of these cases can be found in the example files. A reference manual describing how to use each rubette follows in Section 6.3.

6.2.1 *Random Generation*

The best way to develop a feeling for how a particular counterpoint world sounds is to listen to many of its counterpoints. [Figure 6.3](#), a simplified version of the network found in the subfolder of the examples named `random-generation`, shows how to achieve this with an algorithmic counterpoint composition engine in RUBATO. Given a cantus firmus and composition rules (specified by a counterpoint world), the computer will generate a random counterpoint compatible with these two constraints.

The user needs to specify at least four parameters, represented by white boxes in [Figure 6.3](#).

6.2.1.1 Input score

Use the `MidiFileIn` rubette (Section 6.3.8) for reading the base voice from a given MIDI sequence and load its score into the network. Open the `Properties` dialog by pressing its button, click on the `Browse...` button, navigate to the folder containing the MIDI file with the cantus firmus and select it. Press the `Apply & Hide` button to close the dialog box. You can either use the score provided in the example files called `original-counterpoint.mid`, or prepare one of your own with the software of your choice. The sequence you use should contain a single-voiced sequence of regularly spaced whole notes, as depicted in [Figure 6.4](#).

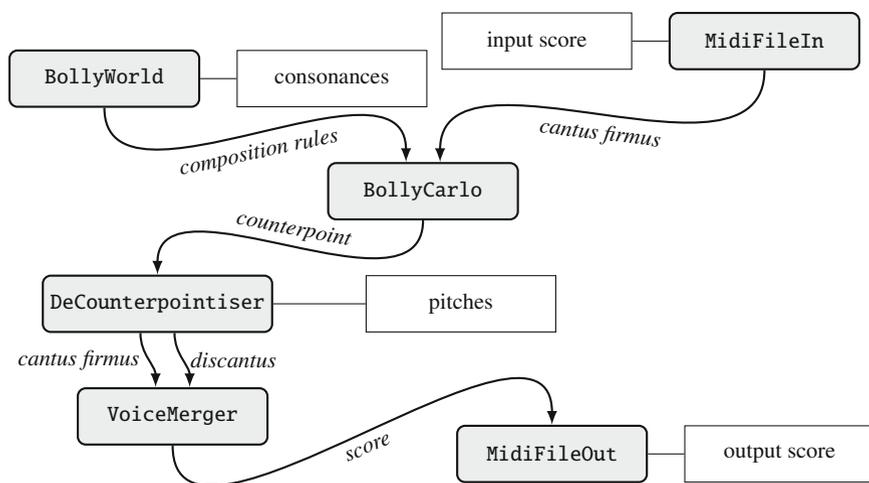


Fig. 6.3 Signal path in a minimal RUBATO network for random counterpoint generation. Shaded boxes represent rubettes (processing units), and curved arrows their connections. White boxes indicate parameters to be set by the user.

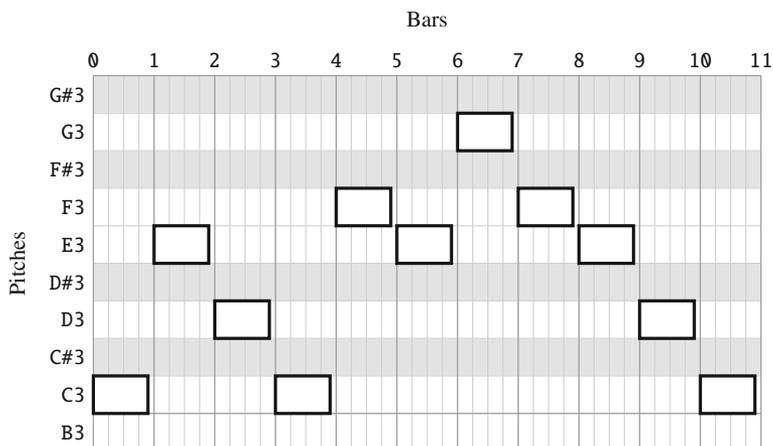


Fig. 6.4 Piano roll representation of a MIDI sequence being used by the counterpoint rubettes. The Counterpointiser rubette will search for notes starting on every bar.

6.2.1.2 Consonances

The **BollyWorld** rubette lets you specify in which counterpoint world you are working. Choose one of the six counterpoint worlds described in Section 3.2 from

the **Strong** dichotomy menu, or build your own set of consonances by pressing the **Custom** radio button appearing on the top of the pane.

6.2.1.3 Trial and Error

Once all rubettes are connected and properly set, launch the **Run** command from the **Network** menu, or press the green arrow in the toolbar. RUBATO will generate a random counterpoint compliant with the chosen rules. The **ScorePlay** rubette allows you to visualize the results in a piano roll display and listen to them. Each time you run the network, a new counterpoint is generated. New settings apply only to the next runs.

6.2.1.4 Pitches

Counterpoint rules hold for equivalence classes, not absolute pitches. Members in each pitch class must be chosen before the actual score can be generated. This module calculates orientations of intervals (sweeping or hanging) and the octave of pitches by default, but you are free to change the proposed values. Open the **DeCounterpointiser** rubette's **Properties** dialog, and drag notes with the mouse in the **Score** tab or change numerical values in the **Table** tab.

6.2.1.5 Random Counterpoint

Once you are happy with a counterpoint, save it into a MIDI file with the help of the **MidiFileOut** rubette, in which you enter the path to the document file that will contain a two-voiced MIDI sequence.

Problem 6.1. *The traditional Fuxian rules and their associated set of consonances is only one among 48 possible choices of consonances. They all share a common structure (up to isomorphism), as stated by the definition of dichotomy classes on page 8. Choose other consonances from the same set. Does it sound different? Can you feel a common structure?*

Problem 6.2. *The network shown in [Figure 6.3](#) implements the traditional counterpoint exercise: add a *discantus* to a given *cantus firmus*. In this case, the *cantus firmus* is always the same since it is read from a static MIDI file. How should the network be altered to also allow variation of the *cantus firmus*—having the computer randomly generate both voices, in a conformant fashion?*

6.2.2 Composition

Instead of letting the computer do all the work, as described in Section 6.2.1, the user can also take control over the composition process. The software suite includes a tool for guiding the composer through the composition rules. It shows which intervals and steps are allowed or forbidden, thus saving the composer from learning complete sets of rules each time he or she steps into a new world. New worlds can be quickly explored by building counterpoints, not just listening. Figure 6.5 shows the necessary components and how to connect them in RUBATO. We briefly describe the parameters that must be set for composing counterpoints in RUBATO.

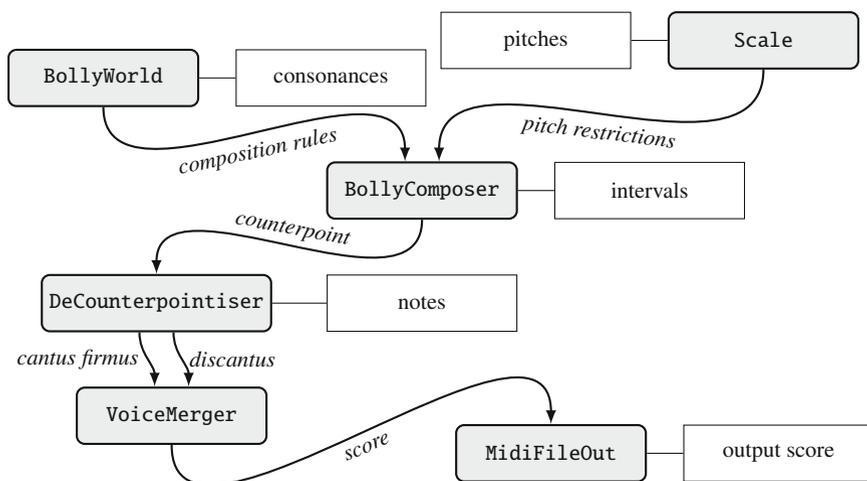


Fig. 6.5 Signal path in a RUBATO network for counterpoint composition. Shaded boxes represent rubettes (processing units), and curved arrows their connections. White boxes indicate parameters to be set by the user.

6.2.2.1 Consonances

Choose the counterpoint world and its associated sets of composition rules; Section 6.3.3 explains how to do this in more detail. Only worlds in the traditional twelve-tone octave division will work because the BollyComposer rubette uses a score representation. You can explore every world described in Chapter 2. There are six classes for the twelve-tone scale, each one containing 48 members, or isomorphic variations. Microtonal counterpoints are not accessible with this tool. A

quite cumbersome workaround would be to build contrapuntal intervals by hand and check their validity using the *AnaBollyser* rubette, see Section 6.2.4.2.

6.2.2.2 Scale

The *Scale* rubette is optional. If connected to the *BollyComposer*, it will further constrain notes to occupy pitches belonging only to the chosen scale. The *Scale* rubette comes in handy for the quick generation of entire scales. Note that since the *BollyComposer* takes a score on its input, you could design your own scale as well, without restricting yourself to octave periodicity. Simply save a cluster of notes into a MIDI file or generate one in *RUBATO*.

6.2.2.3 Intervals

The *BollyComposer* offers a graphical tool for drawing counterpoints directly on the score. The *BollyComposer* has to know in which context it works—the network needs to be run a first time so that the rubette can read this information from its input. It will then be able to inform the user about the correctness of the counterpoint being composed. Run the network each time you want to hear the result in the *ScorePlay* rubette, or save it into a MIDI file using the *MidiFileOut* rubette.

6.2.2.4 Notes

The *DeCounterpointiser* rubette allows you to manually set further note parameters, like loudness or duration.

6.2.2.5 Score

Specify the MIDI file into which to save the counterpoint.

6.2.3 Transformation

A third way of producing counterpoints is to transform existing material. Take advantage of a structure you are happy with, and try to morph it into a new shape, hopefully conserving many of its good properties. The *BollyMorpher* rubette helps you construct such a transformation if the source and target worlds allow it.

Musical material deals with absolute pitch values, but the counterpoint theory of Chapter 2 works with pitch classes. An additional conversion is needed, so that transforming counterpoint occurs in three major steps:

1. Translate absolute pitches of the source material into pitch classes (residue classes modulo 12 in the usual setting) and calculate the interval class between cantus firmus and discantus.
2. Transform the original contrapuntal intervals from the source world into new intervals belonging to the target world.
3. Generate a transformed score by choosing absolute pitches from the target pitch classes.

The first step is a simple matter of encoding. It is not specific to counterpoint and appears as soon as one deals with pitch classes, as in Allen Forte's *set theory* [28]. The last step corresponds to the reverse process but is not trivial since the first step causes a loss of information that has to be recreated. For the composer, it means the freedom of choosing a particular member (i.e., an octave) in each pitch class. The first and last steps are general operations, but the second one is specific to counterpoint. It is also the most difficult part of the operation, the underlying theory of which was laid out in Chapter 5. This section describes how to use the different software tools developed for handling each of these three steps.

6.2.3.1 Source World

The counterpoint rules of the source world are defined with a BollyWorld rubette set to the right world. In the case of a traditional counterpoint, open the parameters by clicking the **Properties** button, select the **Topos of Music** pane at the top of the dialog box, and choose **Fux (82)** from the **Strong dichotomy** menu.

6.2.3.2 Target World

Choose the counterpoint system into which to transform the original counterpoint. If it is the same as the original world (an automorphism), reuse the source BollyWorld rubette. A second BollyWorld rubette is necessary to define a different target world if you wish to transform the musical material into an exotic world. Any of the five other worlds described in Section 3.2 can be found in the **Topos of Music** panel. If you choose to experiment with other representatives of these twelve-tone world classes, or different octave divisions, the **Custom** pane gives more control over the consonances selection, which is fully part of the creative process.

6.2.3.3 Source Counterpoint

Any musical composition properly formatted by the Counterpointiser rubette provides valid material for transformation. Use either a MIDI file or any mean of generating a counterpoint, such as those described in the two previous sections. The only way to build a microtonal counterpoint is by assembling **Simple** and **Constructor** rubettes by hand, as illustrated in [Figure 6.7](#).

6.2.3.4 Interval Mapping

The `BollyMorpher` rubette is not functional at instantiation. Before it is able to guide the user through the mapping process, it needs to know the material to transform (the original counterpoint), and the source and target rules.

After building the network by connecting all rubettes, it is necessary to perform a first run to let the rubette build the morphisms, as described in Chapter 5. Only after this preliminary run will the user be able to choose among the many mappings and eventually produce music—a transformed counterpoint. Enumerating the morphisms can be time-consuming (see Section 5.4.1). But as long as the three inputs do not change, the `BollyMorpher` is ready for experimenting with different mappings, and subsequent runs of the network will generate a transformed counterpoint.

The dialog box depicts the homogeneous digraphs (see Section 4.3.3) of the source world on the left and the target world on the right. Depending on the two worlds and the counterpoint, local and global mappings may or may not exist. An active or inactive state of the radio buttons at the top of the dialog indicates whether a mapping exists. You can specify individual mappings by following the quotient hierarchy: Once a weak mapping has been chosen, the child menus get active and populated, indicating to which homogeneous components a child component can be sent and, eventually, how intervals belonging to a homogeneous component can be mapped.

Another way of constructing a morphism is by using the two buttons at the bottom of the dialog. They allow you either to clean the tree completely from any mapping or to populate it automatically. The new settings take effect only after you press one of the two `Apply` buttons, and changes in the output appear once the network is run again.

6.2.3.5 Target Pitches

The `BollyMorpher` outputs a counterpoint, a data structure based on pitch classes. For the transformed musical piece to be audible or visible, a translation back into absolute pitches is necessary. The `DeCounterpointiser` does it automatically by choosing the pitches nearest to the original composition, but you can also decide which octave to use by using its `Properties` dialog.

6.2.4 Extensions

The three scenarios listed above describe the basic functions of the `BollyFux` rubettes. More complex and interesting results can be produced by combining them with other rubettes or external software. While we offer a couple of suggestions in this last section, the reader is invited to investigate further the different ways of using these tools creatively.

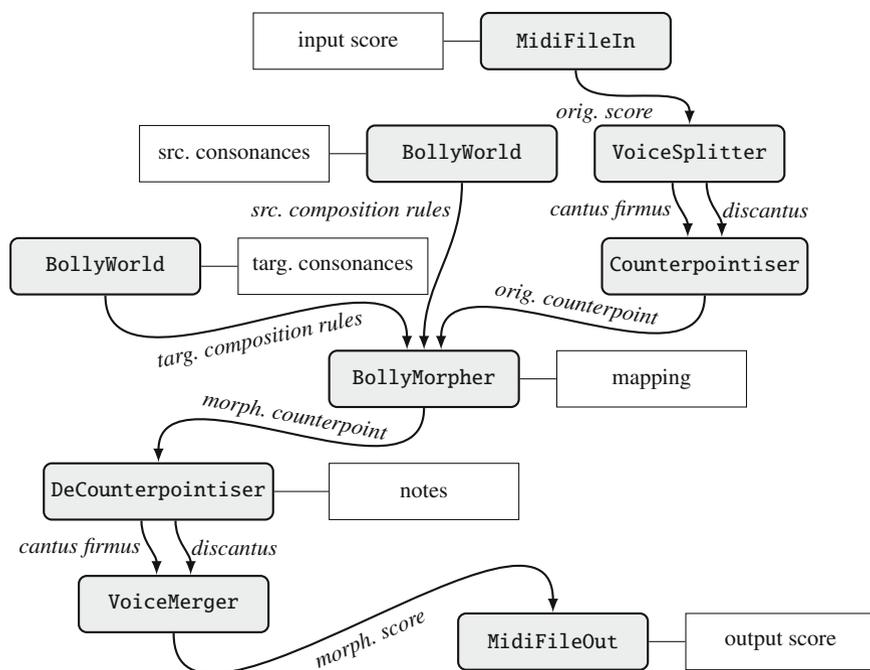


Fig. 6.6 Signal path in a RUBATO network for counterpoint composition. Shaded boxes represent rubettes (processing units), and curved arrows their connections. White boxes indicate parameters to be set by the user.

6.2.4.1 Complexity

The first species counterpoint handled by the rubettes contains only two simultaneous voices, but each note can serve as a basic structure supporting more complex clusters, for example by attaching arpeggios. Try, for instance, to use the *cantus firmus* and the *discantus* as anchor notes for the *MacroScore* rubettes, which are described in [65].

Problem 6.3. Which other means of enriching a basic counterpoint can you imagine? Look at examples from music theory or at tools provided by your favorite music software.

6.2.4.2 Microtonality

The BollyComposer and ScorePlay rubettes are limited to the display of twelve-tone scales, but it is possible to use other octave divisions as well. There is a ScoreToCsound rubette for exporting notes to Csound,³ a music software capable of generating microtonal sounds. RUBATO creates a *score* file you can combine with an appropriate *orchestra* file specifying the virtual instruments that will play the score. Thus, one strategy could be to use a twelve-tone counterpoint and transform it into a six-, eight-, ten-, fourteen- or sixteen-tone counterpoint and listen to the results in Csound.

It is also possible to build directly a microtonal counterpoint, but its constituent parts, the oriented contrapuntal intervals, must be assembled by hand. A network containing the rubettes needed for constructing a minimal counterpoint is shown in Figure 6.7.

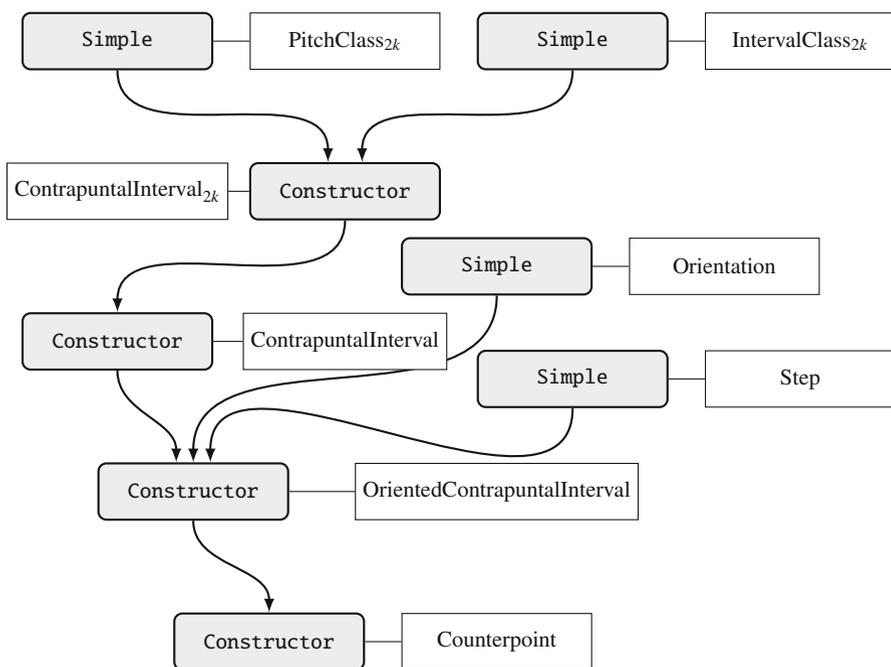


Fig. 6.7 A RUBATO network for building a macro- or microtonal counterpoint with Simple and Constructor rubettes. The *form* or data type, as it appears in each white box, is specified with help of the Properties dialog. Simple rubettes need the additional specification of a value: pitch x and interval i class in $\{0, \dots, 2k-1\}$, sweeping (+1) or hanging (-1) orientation and a step $(0, 1, \dots)$. Only one oriented contrapuntal interval is represented. Repeat the construction for further steps.

³ Csound, can be downloaded from <http://www.csounds.com/> for free and examples can be found on the website and in [15].

6.2.4.3 Timbre

Different instruments can be used to play the counterpoints. We just mentioned Csound, but a twelve-tone counterpoint can be exported also via a standard MIDI file to any sequencer or digital audio workstation. This kind of software offers a lot of virtual instruments for audio rendering and many tools for further editing the score.

6.3 Rubettes

This section describes in more detail the software components dedicated to counterpoint. Every plug-in, called a *rubette*, comes with a series of inputs and/or outputs for communication with other modules. Exchanged data is packaged into a standard format called *forms* and *denotators*. See [57] for the theoretical foundations and [65] for their implementation in RUBATO.

In cases where the data provided by the inputs do not carry all the information necessary for proper operation, additional parameters must be set. Press the *Properties* button and a dialog box will appear presenting all parameters the user can set.

Some rubettes allow the user to visualize their state or data. A *View* button opens a dialog containing such graphical representations. For example, the *Score* rubette, used to play back notes, displays a piano-roll representation like those found in common sequencers, as depicted in [Figure 6.4](#).

The rubettes specifically targeted at handling counterpoint data are gathered inside the *BollyFux* group, under the *Rubettes* list located inside the pane to the right of the *Rubato* window. A couple of other rubettes, whose use is not restricted to counterpoint, may prove to be useful in handling musical material.

6.3.1 Counterpointiser

The *Counterpointiser* rubette constructs a counterpoint out of a musical score. It basically converts absolute pitches into interval classes and computes orientations and distances between the two voices. This encoding does not affect musical material yet; it only extracts information relevant to counterpoint and packages it into a format suitable for further processing.

| Input | Form | Description |
|-------|-----------------|--------------------|
| 1 | StrongDichotomy | Counterpoint world |
| 2 | Score | Cantus firmus |
| 3 | Score | Discantus |

Property

Description

reference pitch:

All pitch classes will be computed as distances relative to the given absolute pitch value, modulo the number of tones.

The rubette needs two kinds of information: an algebraic context for computing pitch classes (reference pitch and number of tones) and the material to encode, stored in the music scores.

The input informs the rubette about the number of tones (which counterpoint world is chosen does not matter here). The reference pitch value tells which note will belong to the 0 class. When pitches have integral values, they correspond to MIDI codes. With a reference pitch set to 60, every C is assigned to class 0, every C# to class 1 and so on. It is possible to have also the scale start on any other pitch. RUBATO further allows microtonal values, measured in fractions. For example, in a 24-tone world $60 + \frac{1}{2}$ would designate a quarter-tone above C3, and pitch classes are not measured in half tones but in quarter tones.

Because the contrapuntal intervals are oriented (the two voices do not play a symmetrical role), the rubette comes with two separated inputs for musical material: one for the cantus firmus, one for the discantus. Therefore, the two voices need to be extracted and routed separately. The `VoiceSplitter` rubette (see Section 6.3.11) can help do this, as long as each voice has been attributed to a distinct MIDI channel in the source sequence.

| Output | Form | Description |
|--------|--------------|--------------------------------------|
| 1 | Counterpoint | An encoded counterpoint |
| 2 | Score | A copy of the original cantus firmus |
| 3 | Score | A copy of the original discantus |

The counterpoint does not hold the entire information characterizing a standard MIDI note. It is just a sequence of pitch and interval classes. Velocity and duration are lost.

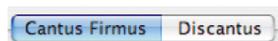
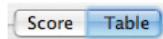
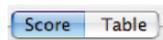
See also Section 6.3.2 for the reverse operation.

6.3.2 *DeCounterpointiser*

The `DeCounterpointiser` reverses the `Counterpointiser`'s operation. It decodes musical material from the format needed for counterpoint processing into a standard musical score. When we use relative pitch classes, we lose part of the information, namely which member of a pitch class was present in the original score. This value has to be provided for the reconstruction of "real" notes, and the `DeCounterpointiser` allows imputation of absolute pitches.

| Input | Form | Description |
|--------------|--------------|--------------------------------------------------|
| 1 | Counterpoint | Counterpoint to decode |
| 2 | Score | Note parameters for the cantus firmus (optional) |
| 3 | Score | Note parameters for the discantus (optional) |

If no input scores are specified, `DeCounterpointiser` maps the class 0 to `C3` by default and follows the sequential order of pitch classes from there.

Property

0



C3



63



0

Description

Choose the score representation of the counterpoint, in a twelve-tone scale.

Notes can be dragged between possible representatives of their classes (by octave jumps or by flipping the contrapuntal interval between its sweeping and hanging orientations).

Complete control over each of the five note parameters can be gained by displaying the **Table** tab, where each value can be specified manually.

There are two subtabs, one for each voice. Each note appears on its own line, ordered by successive onsets.

The relative time at which the note is played: A score starts at 0.0, a beat measures 1.0. Adjust the **Tempo** factor in the **ScorePlay** rubette to change playback speed, see Section 6.3.10. The absolute **Pitch** of a note. Integral values correspond to MIDI conventions and standard tuning; fractional numbers are used in a microtonal context.

Loudness ranges from 0 for quiet notes to 127 for the loudest ones. This parameter is mapped to MIDI velocity.

Voice is an index ranging from 0 to 15. It is mapped to MIDI channels.

Feed the inputs with scores, and the **DeCounterpointiser** will store them to automatically reconstruct notes by reusing the original parameters: nearest absolute pitches and identical onset, duration, and velocity.

Output Form**Description**

| | | |
|---|-------|---------------------|
| 1 | Score | Cantus firmus score |
| 2 | Score | Discantus score |

Use the **VoiceMerger** rubette if you do not need separate scores for the two voices and wish to generate a common one.

See also Section 6.3.1 for the reverse operation.

6.3.3 BollyWorld

Use the BollyWorld rubette to choose a particular counterpoint world. This will set the contrapuntal context in which other rubettes will work. A StrongDichotomy identifies fully a counterpoint world—it contains all the information necessary to generate it.

Output Form

1 StrongDichotomy

Description

The counterpoint world.

Property

Topos of Music

Strong dichotomy: Dur/Bolly (64)

Custom

Octave division: 12

Class: 87

Member: {0,1,2,5,6,9}

2 3 4 5

Multiplicative monoid

Description

You can start with one of the six strong dichotomies described in Chapter 2, or on page 654 of [57].

Numbers correspond to the class indexes found in appendix L on page 1173 and throughout part VII of [57]. A class is instantiated by the representative having the smallest pitch values, except for the Fuxian 82 class, which is represented by the traditional consonance set.

Build your own world.

First, choose your context by setting the number of tones. The value by default is 12; it can be lowered to 6 for macrotonal compositions or raised for microtonal compositions. Values higher than 16 may exhaust the memory capacity of some computers.

Choose the class, or family, of worlds. Each class represents a different contrapuntal structure, with different possibilities left open for composition. Here the numbering of twelve-tone worlds differs slightly from [57], but the ordering is the same.

Each member of the class shows a different permutation of possible consonances. A given interval may or may not be present and play a different role in each combination.

Force intervals to be considered as consonant by selecting them. Only compatible sets remain in the Member menu.

You can require the consonance set to form a multiplicative monoid.

Not all constraints can be fulfilled simultaneously. Incompatible choices may result in an empty `Member` menu. The problem can be solved by loosing constraints until the menu gets populated again. Until a member has been chosen, no dichotomy can be sent to the output.

6.3.4 *BollyCarlo*

Given a base voice and composition rules, the `BollyCarlo` rubette generates a counterpoint by adding a discantus to the cantus firmus. It is a simple constraint-based random generation process, where the counterpoint worlds provide the rules.

| Input | Form | Description |
|--------------|------------------------------|--------------------|
| 1 | <code>StrongDichotomy</code> | Counterpoint world |
| 2 | <code>Score</code> | Cantus firmus |

Rules are specified by the counterpoint world, and the base voice by the cantus firmus. The output counterpoint will contain as many intervals as there are notes in the input score.

| Output | Form | Description |
|---------------|---------------------------|------------------------|
| 1 | <code>Counterpoint</code> | Generated counterpoint |

6.3.5 *BollyComposer*

The `BollyComposer` is a computer-assisted composition tool. Use it to compose a counterpoint from scratch in compliance with the rules of a given counterpoint and scale.

| Input | Form | Description |
|--------------|------------------------------|----------------------------|
| 1 | <code>StrongDichotomy</code> | A counterpoint world |
| 2 | <code>Score</code> | A musical scale (optional) |

If a scale is specified, pitches are further restricted to values belonging to the scale. In this context, a musical scale is basically defined as a list of authorized pitch values covering the entire keyboard. All members of each pitch class should be listed, not just the representative in one octave.

Property

Number of intervals:



Step 0: dissonance (4₀).

Description

Set the number of steps of the composition. Increasing the count will append notes, and decreasing deletes the last notes of each score. Drag notes in the Cantus firmus and Discantus scores to build your composition. If a scale is specified, notes will jump between allowed values. The Status bar informs you whether pitches are compliant with the rules. If this is not the case, the reason is indicated at the bottom of the dialog.

The network needs to be run once before the BollyComposer can be used. When loaded or newly added to the network, the rubette ignores its context (composition constraints), which it will learn from its inputs during this first pass. A second pass is necessary to send the composed counterpoint to the next rubettes in the processing chain.

Output Form

Description

| | | |
|---|--------------|-----------------|
| 1 | Counterpoint | A counterpoint |
| 2 | Score | A cantus firmus |
| 2 | Score | A discantus |

Only pitch values are specified within the BollyComposer. In order to edit other parameters such as loudness and duration, use the DeCounterpointiser (see Section 6.3.2).

6.3.6 BollyMorpher

Use the BollyMorpher rubette to transform a counterpoint from one world into another.

Input Form

Description

| | | |
|---|-----------------|-----------------------|
| 1 | Counterpoint | Original counterpoint |
| 2 | StrongDichotomy | Source world |
| 3 | StrongDichotomy | Target world |

The allowed or forbidden nature of intervals and steps of the original counterpoint is evaluated in the source world, and the contrapuntal structure embedded into the target world.

Property
 Global Local

 Global Local

 $(4,7)_{0|2} \rightarrow (9)_{0|2}$
 $(2,4,7,9)_{0|2}$ $(2,4,7,9)_{0|2}$

It is not possible to Apply partial maps. An error message appears as long as unassigned intervals remain.

Output Form

1 Counterpoint

Description

Press the Global radio button if you wish to morph the entire world, i.e., find a mapping for every interval. This option may not always be available, depending on the selected counterpoint worlds, as discussed in Section 5.4.2.

Press the Local radio button to restrict the mapping to intervals used by the original counterpoint. This option may also be unavailable if the counterpoint uses too many intervals, inducing a subgraph too large to embed into the target world.

Homogeneous graphs of both worlds are depicted on the side areas of the dialog—the source world on the left, the target world on the right. Thick circles represent nodes used by the mapping. Drag the nodes with the mouse to improve the graph display.

The mapping of intervals is located in the central part of the dialog, where the folder tree follows the quotient graph hierarchy. Click on the disclosure triangles to show or hide finer subgraphs. A mapping must be constructed from the broadest (weak components) to the finest assignments (single intervals). Each menu will propose mappings compatible with the assignments already made. If only one choice is left, it will be chosen automatically.

Press the Random button to let the computer construct a random mapping automatically.

Press the Reset button to clear all assignments and start with an empty map.

Description

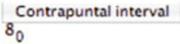
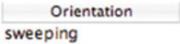
Morphed counterpoint

6.3.7 AnaBollyser

To inspect the validity of a counterpoint, use the AnaBollyser rubette.

| Input | Form | Description |
|-------|-----------------|-------------------------|
| 1 | StrongDichotomy | Counterpoint world |
| 2 | Counterpoint | Counterpoint to analyze |

On pressing the View button, the ordered list of contrapuntal intervals gets displayed. Colors indicate compliance with composition rules: Consonances and allowed steps appear in black, dissonances and forbidden steps in red.

| View | Description |
|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|  | The contrapuntal interval's position in the sequence. Since it needs two intervals to make a step, the sequence starts with step 0 and the second interval resides on step 1. |
|  | The main number indicates the interval class separating the two voices, the subscript value shows the anchor, or cantus firmus class. All numbers are given modulo the number of tones. |
|  | The last column tells whether the interval is hanging or sweeping in the original score. |

6.3.8 Midi File In

The MidiFileIn loads a score from a MIDI sequence.

| Properties | Description |
|-----------------|----------------------------------------------------------------------|
| MIDI input file | Press the Browse button to open the dialog and locate the MIDI file. |

The rubette only reads information relevant for notes. Program changes, sysex, continuous controllers, etc., are discarded.

| Output | Form | Description |
|--------|-------|-------------|
| 1 | Score | Music score |

See also Section 6.3.9 for the reverse operation.

6.3.9 Midi File Out

The MidiFileOut rubette saves a score to a MIDI sequence.

| Input | Form | Description |
|-------|-------|-------------|
| 1 | Score | Music score |

Note that because part of the information is lost during the process, reading from and writing back to a MIDI file may not produce an identical sequence.

| Properties | Description |
|------------------|-----------------------------------------------------------------------------|
| MIDI output file | Press the Browse button to open the dialog and locate the MIDI file. |

See also Section 6.3.8 for the reverse operation, and the `ScoreToCsound` rubette for exporting scores to Csound format.

6.3.10 Score Play

Use the `ScorePlay` rubette to visualize and hear a score.

| Input | Form | Description |
|-------|-------|-------------|
| 1 | Score | Music score |

Notes will be displayed in a piano-roll representation, like in most sequencers. Use this rubette to preview and hear musical material.

| View | Description |
|-------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------|
|  | The piano roll will scroll automatically to display the notes being played. |
|  | Start playback and scrolling of the piano roll from the beginning. |
|  | Pause and resume playback from current position, without going back to the beginning. |
|  | Stop playback and rewind to the beginning of the piano roll. |
|  | Choose which instruments should render each voice. |
|  | Drag the Tempo factor slider to slow down or speed up playback. This come in handy when using whole notes in counterpoints. |

The `ScorePlay` uses the computer's built-in MIDI engine to render MIDI notes. If you happen to use microtonal (non-integer) pitch values, the `ScorePlay` rubette will simply round them to the nearest integer.

6.3.11 Voice Splitter

The VoiceSplitter rubette separates the voices of a music score.

| Input | Form | Description |
|--------------|-------------|--------------------|
| 1 | Score | A music score |

The VoiceSplitter filters all notes from the input score on a voice basis. It does not affect any parameter, but packs notes into a new score and sends it to the corresponding output. If there are no notes for a given voice, the output score will be empty.

| Output | Form | Description |
|---------------|-------------|----------------------------------------|
| 1 | Score | Score containing all notes of voice 0 |
| ⋮ | ⋮ | |
| 16 | Score | Score containing all notes of voice 15 |

See also Section 6.3.12 for the reverse operation.

6.3.12 Voice Merger

The VoiceMerger rubette builds one score out of many.

| Input | Form | Description |
|--------------|-------------|------------------------------------------------------|
| 1 | Score | Score containing all notes that will end on voice 0 |
| ⋮ | ⋮ | |
| 16 | Score | Score containing all notes that will end on voice 15 |

All separate inputs are merged into a single score. Each note will keep its parameters unchanged, except the voice, which will be assigned a value corresponding to the rubette’s input, regardless of any previous setting.

| Output | Form | Description |
|---------------|-------------|--------------------|
| 1 | Score | Music score |

See also Section 6.3.11 for the reverse operation.

Chapter 7

Quasipolarities and Interval Dichotomies

As we have seen in Chapter 2, affine involutive derangements are important for our mathematical theory of counterpoint, because they are the polarities of strong dichotomies in \mathbb{Z}_{2k} . In this chapter we characterize them for future reference.

7.1 Introductory Remarks

Remember first that

$$|\mathrm{GL}(\mathbb{Z}_n)| = \varphi(n)$$

where φ is the Euler totient function. Hence $|\overrightarrow{\mathrm{GL}}(\mathbb{Z}_n)| = n\varphi(n)$.

Let $g = T^u \cdot v \in \overrightarrow{\mathrm{GL}}(\mathbb{Z}_n)$. The function g is an affine involution if and only if

$$v^2 = 1, \tag{7.1}$$

$$u(v+1) = 0. \tag{7.2}$$

Let $n = 2^\alpha p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell}$ be the prime decomposition of n . In [85] it is shown that the solutions of $v^2 \equiv 1 \pmod{2k}$ are given by the simultaneous solutions of the pair of congruences

$$x \equiv 1 \pmod{2a}, \quad x \equiv -1 \pmod{2b} \tag{7.3}$$

if $\alpha = 1, 2$, or the two pairs

$$\begin{aligned} x &\equiv 1 \pmod{2^{\alpha-1}a}, & x &\equiv -1 \pmod{2^b}, \\ x &\equiv 1 \pmod{2a}, & x &\equiv -1 \pmod{2^{\alpha-1}b}, \end{aligned} \tag{7.4}$$

if $\alpha > 2$. Thus, if we know the prime factorization of n , the problem of finding the involutions of $\mathrm{GL}(\mathbb{Z}_n)$ is completely solved.

For g to be a derangement, it is necessary and sufficient that the equation

$$(v-1)x = -u \tag{7.5}$$

has no solutions. From now on v will be a representative of the class v and

$$\sigma(v, n) = \gcd(v-1, n), \quad \tau(v, n) = \gcd(v+1, n);$$

the values of u that satisfy (7.2) are the multiples of

$$u_0 = \frac{n}{\tau(v, n)}.$$

7.2 Characterization of Quasipolarities

Definition 7.1. A quasipolarity is an affine symmetry that it is also an involutive derangement.

Equation (7.5) does not hold if and only if $\sigma(v, n) = \gcd(v-1, n)$ does not divide $-u$ (or, equivalently, it does not divide u). We must mention here that, dealing with an involution, n cannot be odd since every involution acting on a set of even cardinality has a fixed point. Therefore, we will take $n = 2k$ in the following.

By the division algorithm, u must be a multiple of u_0 of form

$$u = \sigma(v, 2k)q + r, \quad 0 < r < \sigma(v, 2k), q \in \mathbb{Z}. \tag{7.6}$$

The following theorem provides a complete characterization of affine quasipolarities. The proof, although not difficult, is not directly related to our discussion. The interested reader may review it in [5].

Theorem 7.1. The affine transformation $g = T^u \cdot v \in \overrightarrow{\text{GL}}(\mathbb{Z}_{2k})$ is aquasipolarity if and only if the following holds:

1. the linear part v is involutive,
2. $u_0 = \frac{2k}{\tau(v, 2k)} < \sigma(v, 2k)$ (in fact, $\sigma(v, 2k) = 2u_0$),
3. u is of form (7.6) with $r = u_0$.

If k is odd, the second condition can be omitted.

Remark 7.1. Note that Lemma 2.1 is a direct corollary of Theorem 7.1 taking $v = -1$, since

$$\sigma(v, 2k) = \gcd(-2, 2k) = 2, \quad \tau(v, 2k) = \gcd(0, 2k) = 2k.$$

Then

$$u_0 = \frac{2k}{\tau(v, 2k)} = \frac{2k}{2k} = 1$$

and thus every integer of the form

$$u = \sigma(v, 2k)q + u_0 = 2q + 1$$

(namely, all the odd integers) are such that $T^u \cdot -1$ is a quasipolarity.

Proposition 7.1. *The isotropy group of an affine involutive derangement $g = T^u \cdot v \in \overrightarrow{\text{GL}}(\mathbb{Z}_{2k})$ under the conjugation action has cardinality $\sigma(v, 2k)\varphi(2k)$. Equivalently, there are $\frac{2k}{\sigma(v, 2k)}$ elements in the orbit of g under the conjugation action.*

Proof. Let $h = T^t \cdot s \in \overrightarrow{\text{GL}}(\mathbb{Z}_{2k})$. We have

$$g' = h \circ g \circ h^{-1} = (T^t \cdot s) \circ (T^u \cdot v) \circ (T^{-s^{-1}t} \cdot s^{-1}) = T^{t(1-v)+su} \cdot v. \quad (7.7)$$

The members of the stabilizer of g are those whose parameters satisfy

$$t(1 - v) = u(1 - s).$$

This equation has a solution for each $s \in \text{GL}(\mathbb{Z}_{2k})$ since $\sigma(v, 2k) = \text{gcd}(1 - v, 2k)$ divides $u(1 - s)$ because $1 - s$ is even.

In particular, the equation above has exactly $\sigma(v, 2k)$ different solutions. Since this occurs for each $s \in \text{GL}(\mathbb{Z}_{2k})$, the cardinality of the stabilizer of g is

$$|\overrightarrow{\text{GL}}(\mathbb{Z}_{2k})_g| = \sigma(v, 2k)\varphi(2k).$$

This means that each orbit has

$$\frac{|\overrightarrow{\text{GL}}(\mathbb{Z}_{2k})|}{\sigma(v, 2k)\varphi(2k)} = \frac{2k}{\sigma(v, 2k)}$$

elements. □

Corollary 7.1. *The group $\overrightarrow{\text{GL}}(\mathbb{Z}_{2k})$ acts transitively on the set of all affine involutive derangements with fixed linear parts.*

7.3 Calculation of Strong Dichotomies

Here we take a slightly more general approach to the enumeration of strong dichotomies. First, let G be a subgroup of the symmetric group of a finite set S . The group G also acts on the powerset 2^S of S , with an action defined by

$$\begin{aligned} \cdot : G \times 2^S &\longrightarrow 2^S, \\ (g, A) &\longmapsto gA := \{ga : a \in A\}. \end{aligned}$$

Two sets $A, B \in 2^S$ are said to be *isomorphic* if there exists $g \in G$ such that $A = gB$. We will always consider a set S of even cardinality and the subset $\text{MiD}(S)$ of its power set, whose elements we called (marked) *dichotomies* in Chapter 2.

Definition 7.2. *The dichotomy $D \in \text{MiD}(S)$ is:*

1. G -rigid when $gD = D$ for $g \in G$ implies $g = e$,
2. G -autocomplementary if there exists a $p \in G$ (called autocomplementarity function) such that

$$pD = D^c := S \setminus D,$$

3. G -strong if it is G -rigid and G -autocomplementary.

As before, we may note the following:

1. If a dichotomy $D \in \text{MiD}(S)$ is G -strong and $p \in G$ is such that $pD = D^c$, then p is unique. We call it the *polarity* of D . For any $g \in G$, the set gD is also G -strong (with polarity gpg^{-1}).
2. Since S is finite, any polarity is an involution that does not fix points of S , i.e., it is an involutive derangement.

Naturally, we will call an involutive derangement $q \in G$ *quasipolarity*. We can proceed again decomposing q in 2-cycles, so we obtain a dichotomy $U_q \in \text{MiD}(S)$ that contains exactly one element from each cycle, and it is such that U_q is autocomplementary and $qU_q = U_q^c$. The following lemma, which is useful for the generation of strong dichotomies, is proved in [5].

Lemma 7.1. *Let G_q be the isotropy group of a quasipolarity $q \in G$ under the conjugation action. Define*

$$M_q = \{D \in \text{MiD}(S) : qD = S \setminus D\},$$

which is the set of dichotomies whose autocomplementarity function is q . If $g \in G_q$ and $h \in G$, then $gM_q = M_q$ and $M_{hgh^{-1}} = hM_q$.

Note that

$$M_q = \{A \cup B : A \subseteq U_q, B = (S \setminus U_q) \setminus qA\},$$

where the last equality follows from

$$qA = (S \setminus U_q) \setminus A \quad \text{and} \quad q((S \setminus U_q) \setminus qA) = U_q \setminus A,$$

and also that $|M_q| = 2^{|U_q|} = 2^{|S|/2}$.

Let T be a traversal of $\text{MiD}(S)/G$ (i.e., a set consisting of exactly one element from each G -orbit on $\text{MiD}(S)/G$) and $\mathcal{R}_G \subseteq T$ a subset such that all of its members are G -strong. For any $M_1, M_2 \in T$, it is impossible that $M_1 \subseteq M_2$, for then $M_1 = M_2$ and they would represent the same orbit. Thus T and \mathcal{R}_G are antichains. In particular, \mathcal{R}_G is a *complement-free* antichain, since $D \in \mathcal{R}_G$ implies that $D^c \notin \mathcal{R}_G$. Indeed, if $D_1 \cap D_2 = \emptyset$, it would imply that $D_1 = D_2^c = pD_2$ for some $p \in G$, hence D_1 would be in the orbit of D_2 .

Definition 7.3. *Let G be a subgroup of the symmetric group of a finite set S , with $|S|$ even. A G -strong antichain is a subset \mathcal{R}_G of a traversal T of the orbit set $\text{MiD}(S)/G$, such that all its members are G -strong.*

It is obvious that the cardinality of a G -strong antichain \mathcal{R}_G is not greater than the number of $(|S|/2)$ -subsets of S :

$$|\mathcal{R}_G| \leq \binom{|S|}{|S|/2}.$$

Being \mathcal{R}_G a complement-free antichain, a theorem by Purdy [9, p. 139] tells us that

$$|\mathcal{R}_G| \leq \binom{|S|}{|S|/2 - 1},$$

and using the Erdős-Ko-Rado theorem [26, theorem 1], we can improve this slightly:

$$|\mathcal{R}_G| \leq \binom{|S| - 1}{|S|/2 - 1}.$$

These upper bounds, however, do not make full use of the structure of \mathcal{R}_G derived from the action of G on S . In order to exploit it, first note that G -strong dichotomies have orbits of maximum cardinality, namely $|G|$.

The number of non-isomorphic G_q -strong dichotomies for a given quasipolarity q is bounded by

$$|\mathcal{R}_{G_q}| \leq \frac{|M_q|}{|G_q|} = \frac{2^{|S|/2}}{|G_q|}, \tag{7.8}$$

since, by Lemma 7.1, the set M_q of dichotomies whose autocomplementary function is q coincide with $M_{q'}$, for any conjugate q' of q .

For \mathbb{Z}_{2k} , note that by Theorem 7.1 and the proof of Proposition 7.1, the orbits under the conjugation action of quasipolarities are determined by their linear parts. This means that the orbits under conjugation of quasipolarities $f_1 = T^{u_1} \cdot v_1$ and $f_2 = T^{u_2} \cdot v_2$ coincide if and only if $v_1 = v_2$.

Consider the decomposition in transpositions of the quasipolarity p ,

$$(x_1 \ p(x_1))(x_2 \ p(x_2)) \cdots (x_k \ p(x_k));$$

so we can define $C = \{x_i\}_{i=1}^k$ such that $(C/p(C))$ is an autocomplementary interval dichotomy. Send an autocomplementary interval dichotomy (X/Y) to a subset of C via the function

$$\begin{aligned} \kappa_{p,C} : \mathcal{A}_p &\longrightarrow \wp(C), \\ X &\longmapsto X \cap C, \end{aligned}$$

and we map a subset $D \subseteq C$ to the autocomplementary interval dichotomy defining

$$\begin{aligned} \theta_{p,C} : \wp(C) &\longrightarrow \mathcal{A}_p, \\ D &\longmapsto D \cup p(C \setminus D). \end{aligned}$$

Obviously $\theta_{p,C}$ and $\kappa_{p,C}$ are mutual inverses. By Lemma 7.1, we can find all the strong interval dichotomies calculating, for every subset $D \in \wp(C)$, the isotropy group

of the marked dichotomy $\theta(D)$ and eliminating those non-rigid $\theta(D)$. Moreover, if we examine a particular D , then it is not necessary to examine $C \setminus D$ since $\theta(C \setminus D) = p(D)$, which means that $C \setminus D$ is in the orbit of D .

For the following, we define the bijection

$$\omega : \wp C \longrightarrow \{0, 1, \dots, 2^k\},$$

$$D \longmapsto \sum_{i=1}^k \chi_D(x_i) 2^{i-1}$$

where $C = \{x_i\}_{i=1}^k$ is an autocomplementary dichotomy in \mathbb{Z}_{2k} and χ_D is the characteristic function of the set D . Note that $\omega(D^c) = 2^{k-1} - \omega(D)$, so $\{\omega^{-1}(i)\}_{i=0}^{2^{k-1}-1}$ consists of all the subsets of C such that no two are complementary. We can write now the “naive” algorithm to calculate marked classes of strong dichotomies.

Algorithm 7.1. We consider $\overrightarrow{\text{GL}}(\mathbb{Z}_{2k}) = \{g_1 = T^0 \cdot 1, g_2, \dots, g_{|\overrightarrow{\text{GL}}(\mathbb{Z}_{2k})|}\}$.

Require: A quasipolarity p in \mathbb{Z}_{2k} .

Ensure: The set F are the representatives of the classes of strong dichotomies with polarity p .

```

1  C ← ∅, F ← ∅;
2  foreach x ∈ ℤ2k {
3      if x, p(x) ∉ C {
4          C ← C ∪ {x};
5      }
6  }
7  foreach 0 ≤ i < 2k-1 {
8      b ← 1, X ← θ(ω-1(i)), ℓ ← 2;
9      while ℓ ≤ | $\overrightarrow{\text{GL}}(\mathbb{Z}_{2k})$ | and b ≠ 0 {
10         if gℓX = X or gℓX ∈ F {
11             b ← 0;
12         }
13         ℓ ← ℓ + 1;
14     }
15     if b = 0 {
16         F ← F ∪ {X};
17     }
18 }
```

Table 7.1 shows how many classes of strong dichotomies there are in \mathbb{Z}_{2k} , $2 \leq 2k \leq 42$, calculated using Algorithm 7.1.

We have left for the end of this chapter the deduction of a non-trivial upper bound for the number of strong dichotomy classes. We begin noting that every G -strong dichotomy is a G_q -strong dichotomy because G_q is a subgroup of G , hence $|\mathcal{R}_G| \leq \sum_{[q] \in \mathcal{Q}} |\mathcal{R}_{G_q}|$, where \mathcal{Q} is the set of conjugacy classes of the quasipolarities of G . Now summing (7.8) over \mathcal{Q} , we prove the following result.

| | | | | | | | | | | | | | | | | | |
|-------|------|-------|-------|-------|--------|--------|--------|--------|-------|----|----|-----|-----|-----|------|------|------|
| $2k$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | $2k$ | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 |
| N_D | 1 | 0 | 1 | 1 | 3 | 6 | 9 | 15 | N_D | 40 | 90 | 105 | 359 | 355 | 1092 | 3007 | 2152 |
| $2k$ | 34 | 36 | 38 | 40 | 42 | 44 | 46 | 48 | | | | | | | | | |
| N_D | 4305 | 17826 | 15267 | 48549 | 130839 | 170820 | 198753 | 780645 | | | | | | | | | |

Table 7.1 The number of strong dichotomies for $2 \leq 2k \leq 48$.

Theorem 7.2. Let G be a subgroup of the symmetric group of the set S (where $|S|$ is an even number) and \mathcal{R}_G be a G -strong antichain. Then

$$|\mathcal{R}_G| \leq \sum_{[q] \in Q} \frac{2^{|S|/2}}{|G_q|}, \tag{7.9}$$

where Q is the set of conjugacy classes of quasipolarities of G .

Remark 7.2. Observe that if we have better estimations of the number of quasipolarities of G and of the sizes of their isotropy groups, we can improve the bound of Theorem 7.2. But even with those refinements, (7.9) might be far from optimal.

For a $\overrightarrow{\text{GL}}(\mathbb{Z}_{2k})$ -strong antichain, we have by Proposition 7.1

$$|\overrightarrow{\text{GL}}(\mathbb{Z}_{2k})_g| = \sigma_{2k}(v)\varphi(2k) \geq 2\varphi(2k)$$

using the involution $v = -1$. The number of conjugacy classes of quasipolarities is bounded by $\varphi(2k)$, thus Theorem 7.2 gives us

$$|\mathcal{R}_{\overrightarrow{\text{GL}}(\mathbb{Z}_{2k})}| \leq \varphi(2k) \frac{2^k}{2\varphi(2k)} = 2^{k-1}.$$

This estimate can be improved further with a better bound on the number of involutions of $\text{GL}(\mathbb{Z}_{2k})$, a beautiful open problem.

Chapter 8

Towers of Counterpoint

The aim of this chapter is to introduce the notion of *tower of counterpoint*, which arises from the progressive embedding of a microtonal setup in another by subdividing (and sometimes shifting) tones. We show that there exist infinite towers of counterpoint, and in particular one that contains an isomorphic instance of the classical consonance-dissonance dichotomy. It is also proven that one of these infinite towers is such that its limit is dense in the octave, and its polarity can be extended continuously with respect to the standard topology of S^1 .

8.1 The Category of Strong Dichotomies

Let $\mathcal{A}_1 = (X_1/Y_1)$ and $\mathcal{A}_2 = (X_2/Y_2)$ two strong dichotomies in ambient spaces \mathbb{Z}_{2k_1} and \mathbb{Z}_{2k_2} , respectively. A morphism between these dichotomies is a morphism of modules $\phi : \mathbb{Z}_{2k_1} \rightarrow \mathbb{Z}_{2k_2}$ such that

$$\phi(X_1) \subseteq X_2, \phi(Y_1) \subseteq Y_2$$

and the following square commutes

$$\begin{array}{ccc} \mathbb{Z}_{2k_1} & \xrightarrow{\phi} & \mathbb{Z}_{2k_2} \\ p_{\mathcal{A}_1} \downarrow & & \downarrow p_{\mathcal{A}_2} \\ \mathbb{Z}_{2k_1} & \xrightarrow{\phi} & \mathbb{Z}_{2k_2}, \end{array}$$

where $p_{\mathcal{A}_1}$ and $p_{\mathcal{A}_2}$ are the respective polarities of \mathcal{A}_1 and \mathcal{A}_2 . Defined in this way, the morphisms of strong dichotomies become a category denoted with $\mathfrak{D}\mathfrak{F}$.

We are particularly interested in a morphism of dichotomies where

$$\begin{aligned} \phi : \mathbb{Z}_k &\longrightarrow \mathbb{Z}_{2k}, \\ x &\longmapsto 2x \end{aligned}$$

is the canonical injection, and more specifically in the dichotomies where $k = 2^n \cdot 3$ for some $n > 0$.

To show that such dichotomies exists, first we prove that there exist certain quasipolarities in their underlying spaces.

Proposition 8.1. *The affine map*

$$T^{2^{n-1}} \cdot (4^{\lceil \frac{n}{2} \rceil} + 1) \in \overrightarrow{\text{GL}}(\mathbb{Z}_{3 \cdot 2^n})$$

is a quasipolarity.

Proof. We begin by defining $v = 4^{\lceil \frac{n}{2} \rceil} + 1$ and noting that

$$\begin{aligned} v^2 &= 4^{2\lceil \frac{n}{2} \rceil} + 2 \cdot 4^{\lceil \frac{n}{2} \rceil} + 1 \\ &= 2^{4\lceil \frac{n}{2} \rceil} + 2^{2\lceil \frac{n}{2} \rceil + 1} + 1 \\ &= 2^{2\lceil \frac{n}{2} \rceil + 1} (2^{2\lceil \frac{n}{2} \rceil - 1} + 1) + 1 \\ &\equiv 1 \pmod{3 \cdot 2^n} \end{aligned}$$

since $2^{2\lceil \frac{n}{2} \rceil - 1} \equiv 2 \pmod{3}$ and $2\lceil \frac{n}{2} \rceil + 1 > n$. Now let

$$\begin{aligned} \sigma &= \gcd(v - 1, 3 \cdot 2^n) \\ &= \gcd(4^{\lceil \frac{n}{2} \rceil}, 3 \cdot 2^n) \\ &= 2^n, \\ \tau &= \gcd(v + 1, 3 \cdot 2^n) \\ &= \gcd(4^{\lceil \frac{n}{2} \rceil} + 2, 3 \cdot 2^n) \\ &= \gcd(2(2^{2\lceil \frac{n}{2} \rceil - 1} + 1), 3 \cdot 2^n) \\ &= 2\gcd(2^{2\lceil \frac{n}{2} \rceil - 1} + 1, 3 \cdot 2^{n-1}) \\ &= 3 \cdot 2. \end{aligned}$$

Thus

$$u = \frac{3 \cdot 2^n}{\tau} = \frac{3 \cdot 2^n}{3 \cdot 2} = 2^{n-1} < \sigma = 2^n.$$

By the criterion given in Theorem 7.1 the map

$$T^u \cdot v = T^{2^{n-1}} \cdot (4^{\lceil \frac{n}{2} \rceil} + 1)$$

is a quasipolarity. □

Proposition 8.2. *The square*

$$\begin{array}{ccc}
 \mathbb{Z}_{3 \cdot 2^n} & \xrightarrow{2} & \mathbb{Z}_{3 \cdot 2^{n+1}} \\
 T^{2^{n-1}} \cdot (4^{\lceil \frac{n}{2} \rceil + 1}) \downarrow & & \downarrow T^{2^n} \cdot (4^{\lceil \frac{n+1}{2} \rceil + 1}) \\
 \mathbb{Z}_{3 \cdot 2^n} & \xrightarrow{2} & \mathbb{Z}_{3 \cdot 2^{n+1}}
 \end{array}$$

is commutative.

Proof. On the one hand we have

$$\begin{aligned}
 2 \circ T^{2^{n-1}} \cdot (4^{\lceil \frac{n}{2} \rceil + 1}) &= T^{2^n} \cdot (2 \cdot 4^{\lceil \frac{n}{2} \rceil}) \\
 &= T^{2^n} \cdot (2^{2\lceil \frac{n}{2} \rceil + 1})
 \end{aligned}$$

while on the other hand

$$\begin{aligned}
 T^{2^n} \cdot (4^{\lceil \frac{n}{2} \rceil + 1}) \circ 2 &= T^{2^n} \cdot (2 \cdot 4^{\lceil \frac{n+1}{2} \rceil}) \\
 &= T^{2^n} \cdot (2^{2\lceil \frac{n+1}{2} \rceil + 1}).
 \end{aligned}$$

The numbers $2\lceil \frac{n+1}{2} \rceil + 1$ and $2\lceil \frac{n}{2} \rceil + 1$ differ by 0 or 2. In the latter case we have

$$\begin{aligned}
 2^{2\lceil \frac{n+1}{2} \rceil + 1} - 2^{2\lceil \frac{n}{2} \rceil + 1} &= 2^{2\lceil \frac{n}{2} \rceil + 1} (2^2 - 1) \\
 &= 2^{2\lceil \frac{n}{2} \rceil + 1} \cdot 3 \\
 &\equiv 0 \pmod{3 \cdot 2^{n+1}}.
 \end{aligned}$$

Consequently, the square commutes. □

8.2 Towers of Counterpoint

The main result of this chapter concerns the existence of a very special infinite directed system of strong dichotomies. In order to construct them, we first need a result about embeddings of strong dichotomies. For the essential definitions of graph theory in the following proofs, see the mathematical appendix, Section A.2.

Lemma 8.1. *Let n be an even number and (X_n/Y_n) a strong dichotomy in \mathbb{Z}_n with polarity $p_n \in \overrightarrow{\text{GL}}(\mathbb{Z}_n)$. For any conjugate p' of p_n there exists an autocomplementary dichotomy (V/V^c) with autocomplementary function p' such that it does not belong to the orbit of X_n nor is it invariant under the antipodal morphism*

$$\begin{aligned}
 T^{n/2} : \mathbb{Z}_n &\longrightarrow \mathbb{Z}_n, \\
 x &\longmapsto x + n/2.
 \end{aligned}$$

Proof. We begin by observing the following inequalities:

$$\begin{aligned} n\varphi(n) < n^2 &\leq \begin{cases} 2^{\frac{n}{2}}, & n > 15, \\ 2^{\frac{n}{4}} - 1, & n > 43, \end{cases} \\ n\varphi(n) < 2^{\frac{n}{2}}, & \quad n = 12, 14, \\ n\varphi(n) < 2^{\frac{n}{4}} - 1, & \quad n = 36, 40. \end{aligned}$$

The orbit of X_n has $n\varphi(n)$ elements, while the number of autocomplementary dichotomies with autocomplementary function p' is $2^{\frac{n}{2}}$. Thus, it is certain that for $n > 12$, there exists at least one dichotomy V that does not belong to the orbit of X_n and $p'(V) = V^c$.

Now we prove that at least one of these V is not invariant under the action of the antipodal morphism. Consider the graph H with \mathbb{Z}_n as vertices and edges $\{x, T^{\frac{n}{2}} \cdot (x)\}$. Color the edges of H *black* if its two vertices belong to V , *white* if its two vertices do not belong to V , and *gray* otherwise. Denote respectively with N_B^V, N_W^V, N_G^V the number of black, white, and gray edges defined by V . Note that (or see [42])

$$2N_B^V + N_G^V = 2N_W^V + N_G^V = \frac{n}{2}$$

and thus $N_B^V = N_W^V$.

If $\frac{n}{2}$ is odd, the equation above implies N_G^V is odd, so $N_G^V \geq 1$. Therefore, at least one element of V is mapped to the complement of V under the antipodal map.

If $\frac{n}{2}$ is even, then N_G^V is even. Suppose $N_G^V = 0$ and thus $N_B^V = N_W^V = \frac{n}{4}$.

In order to find a bijection f between black and white edges, first add to H the edges $\{x, p'(x)\}$ to obtain H' . These edges are not already in H , for it has no gray edges. All the vertices in H' are of degree 2 because any vertex has as neighbors exactly one vertex of its same color (defined by the antipodal map) and exactly one of the opposite color (defined by p') and there are no loops (both p' and the antipodal maps are involutions and do not have fixed points). Hence H' is Eulerian, so it can be decomposed in k cycles. Such cycles are disjoint in the vertices, because otherwise some vertex of H' would have degree more than 2.

In each cycle, no two edges of the same color are adjacent because each of them is defined by the antipodal function. Also no two edges of the same color can be connected by an edge of form $\{x, p'(x)\}$, for this means that both x and $p'(x)$ belong to V or both x and $p'(x)$ belong to V^c , an impossibility by the definition of p' . From this we deduce that in each cycle black, and white edges alternate and are equal in number.

Now endow each cycle with an arbitrary but fixed orientation and order linearly its white and black edges accordingly, such that the least element of the ordering is the black edge with the vertex with the smallest representative. Thus, for the j th cycle that has r_j black edges we have

$$b_{1,j} < w_{1,j} < b_{2,j} < w_{2,j} < \cdots < b_{r_j,j} < w_{r_j,j}.$$

Define f by $f(b_{i,j}) = w_{i,j}$. So defined, f is a bijection, because every black edge belongs to exactly one cycle and its successor in the corresponding order is white and uniquely defined. Observe that the edge $b_{i,j}$ has a vertex x such that $p'(x)$ is in $f(b_{i,j})$. Therefore if we replace x by $p'(x)$, then we obtain a new dichotomy V' such that the $N_G^{V'} = 2$.

Since initially there are $\frac{n}{4}$ black edges, we can construct at least $2^{\frac{n}{4}} - 1$ different autocomplementary dichotomies distinct from V that define at least two gray edges (using all the subsets of the black edges). So, for $n > 35$ there exists at least one of them outside the orbit of X_n .

The cases $n = 6, 8, 10, 12, 16, 20, 24, 28, 32$ are not covered by the previous reasoning, but we have the following dichotomies:

- $\mathbb{Z}_6 \supseteq V = \{1, 2, 3\} + k, p' = T^{2k+1} \cdot 5,$
- $\mathbb{Z}_8 \supseteq V = \{1, 2, 3, 4\} + k, p' = T^{2k+1} \cdot 7,$
- $\mathbb{Z}_{10} \supseteq V = \{1, 2, 3, 4, 5\} + k, p' = T^{2k+1} \cdot 9,$
- $\mathbb{Z}_{12} \supseteq V = \{2, 4, 5, 6, 7, 9\} + 2k, p' = T^{4k+2} \cdot 5,$
- $\mathbb{Z}_{12} \supseteq V = \{1, 2, 3, 4, 5, 6\} + k, p' = T^{2k+1} \cdot 11,$
- $\mathbb{Z}_{16} \supseteq V = \{0, 1, 2, 3, 4, 5, 6, 7\}, p' = T^8 \cdot 1,$
- $\mathbb{Z}_{16} \supseteq V = \{1, 2, 3, 4, 5, 6, 7, 8\} + k, p' = T^{2k+1} \cdot 15.$
- $\mathbb{Z}_{20} \supseteq \{0, 1, 2, 4, 9, 13, 15, 16, 17, 18\}, p' = T^{10} \cdot 1.$
- $\mathbb{Z}_{20} \supseteq \{0, 6, 7, 8, 9, 10, 11, 17, 18, 19\} + 2k, p' = T^{2+4k} \cdot 9.$
- $\mathbb{Z}_{20} \supseteq \{0, 3, 6, 7, 9, 12, 13, 15, 16, 19\} + 5k, p' = T^{5+10k} \cdot 11.$
- $\mathbb{Z}_{20} \supseteq \{1, 3, 10, 12, 13, 14, 15, 16, 17, 19\} + k, p' = T^{1+2k} \cdot 19.$
- $\mathbb{Z}_{24} \supseteq \{0, 1, 2, 4, 9, 10, 11, 15, 17, 18, 19, 20\}, p' = T^{12} \cdot 1.$
- $\mathbb{Z}_{24} \supseteq \{0, 1, 4, 5, 8, 9, 12, 16, 17, 20, 21, 22\} + 3k, p' = T^{3+6k} \cdot 7.$
- $\mathbb{Z}_{24} \supseteq \{0, 1, 2, 3, 5, 6, 9, 11, 12, 15, 20, 22\} + 4k, p' = T^{4+8k} \cdot 17.$
- $\mathbb{Z}_{24} \supseteq \{0, 4, 5, 6, 7, 8, 11, 12, 15, 16, 22, 23\} + k, p' = T^{1+2k} \cdot 23.$
- $\mathbb{Z}_{28} \supseteq \{0, 2, 4, 5, 7, 10, 12, 15, 17, 20, 22, 23, 25, 27\}, p' = T^{14} \cdot 1.$
- $\mathbb{Z}_{28} \supseteq \{1, 2, 6, 7, 8, 10, 11, 13, 14, 18, 21, 25, 26, 27\} + 2k, p' = T^{2+4k} \cdot 13.$
- $\mathbb{Z}_{28} \supseteq \{0, 1, 2, 5, 10, 11, 13, 14, 15, 16, 18, 19, 24, 27\} + 7k, p' = T^7 \cdot 15 + 14k.$
- $\mathbb{Z}_{28} \supseteq \{2, 4, 5, 6, 9, 11, 12, 13, 14, 19, 21, 22, 26\} + k, p' = T^{1+2k} \cdot 27.$
- $\mathbb{Z}_{32} \supseteq \{0, 1, 3, 4, 6, 7, 8, 9, 10, 12, 18, 21, 27, 29, 30, 31\}, p' = T^{16} \cdot 1.$
- $\mathbb{Z}_{32} \supseteq \{0, 2, 7, 8, 9, 10, 15, 17, 19, 20, 21, 22, 27, 28, 29, 30\} + k, p' = T^{1+2k} \cdot 31.$

All of these are not strong and not invariant under their respective antipodal maps.

□

Proof (An alternative proof). For the case $N_G^V = 0$, there is another argument to establish the bijective function between the set $B = \{b_1, \dots, b_{\frac{n}{4}}\}$ of black edges and the set $W = \{w_1, \dots, w_{\frac{n}{4}}\}$ of white edges. Let H' be the graph with vertices $B \cup W$ and edges defined by

$$e = \{x, y\} \in H' \quad p'(x) \cap y \neq \emptyset.$$

All the vertices of H' are of degree 1 or 2, since for an edge x in H we have that $p'(x)$ intersects either one or two edges of H . This happens because every vertex of H belongs to some edge of H and these are pairwise disjoint (the antipodal map being an involution).

Let H'' be the subgraph of H' spanned by the vertices of degree 2 and F' the subgraph of H' spanned by the vertices of degree 1. No edge of H'' crosses from H' to F' , because if a vertex y of F' is adjacent to a vertex x of H'' , then either $p'(x) = y$ or $p'(y) = x$ and the involutivity of p' implies $x \in F'$.

By a theorem of graph theory (see [22, p. 37]), H'' has a 1-factor F , i.e., a 1-regular spanning subgraph. Clearly F' is a 1-factor of itself, hence $F \cup F'$ is a 1-factor of H' .

Now we can define the bijection f by sending b_i to its neighbor in $F' \cup F$. By construction of H' , at least one vertex x of b_i is mapped by p' to a vertex of $p'(x) \in f(b_i)$. Replacing x by $p'(x)$ in the dichotomy V yields a dichotomy V' that induces a new coloring in H such that $N_G^{V'} = 2$. The rest of the proof now goes the same. \square

Lemma 8.2. *Let n be an even number. Suppose there exist quasipolarities $p_n = T^u \cdot v$ and $p_{2n} = T^r \cdot w$ such that the square*

$$\begin{array}{ccc} \mathbb{Z}_n & \xrightarrow{2} & \mathbb{Z}_{2n} \\ p_n \downarrow & & \downarrow p_{2n} \\ \mathbb{Z}_n & \xrightarrow{2} & \mathbb{Z}_{2n} \end{array}$$

commutes. Then $p' = T^{u + \frac{w-1}{2}} \cdot v$ is a quasipolarity in \mathbb{Z}_n .

Proof. Using Theorem 7.1, we know that

$$\begin{aligned} u &= \gcd(v-1, n)q + \frac{n}{\gcd(v+1, n)}, \\ r &= \gcd(w-1, 2n)q' + \frac{2n}{\gcd(w+1, 2n)}, \\ 2 \frac{n}{\gcd(v+1, n)} &= \gcd(v-1, n), \\ 2 \frac{2n}{\gcd(w+1, 2n)} &= \gcd(w-1, 2n). \end{aligned}$$

Observe that because of the commutativity hypothesis, $r \equiv 2u \pmod{2n}$ and $v - w = kn$ for some integer k ; thus

$$\gcd(v \pm 1, n) = \gcd(v - kn \pm 1, n) = \gcd(w \pm 1, n).$$

If $2\gcd(w+1, n) = \gcd(w+1, 2n)$, then

$$\begin{aligned}
\gcd(w-1, 2n) &= 2 \frac{2n}{\gcd(w+1, 2n)} \\
&= 2 \frac{n}{\gcd(w+1, n)} \\
&= 2 \frac{n}{\gcd(v+1, n)} \\
&= \gcd(v-1, n) = \gcd(w-1, n).
\end{aligned}$$

Also

$$\begin{aligned}
2u = r &= \gcd(w-1, 2n)q' + \frac{2n}{\gcd(w+1, 2n)} \\
&= \gcd(w-1, n)q' + \frac{n}{\gcd(w+1, n)} \\
&= \gcd(v-1, n)q' + \frac{n}{\gcd(v+1, n)},
\end{aligned}$$

hence $u = \gcd(v-1, n)(q - q')$, implying $\gcd(v-1, n) \mid u$. But this is impossible because of Theorem 7.1.

If $\gcd(v+1, n) = \gcd(w+1, n) = \gcd(w+1, 2n)$, then

$$\gcd(v-1, n) = 2 \frac{n}{\gcd(v+1, n)} = \frac{2n}{\gcd(w+1, n)} = \frac{2n}{\gcd(w+1, 2n)}$$

and

$$\begin{aligned}
r &= \gcd(w-1, 2n)q' + \frac{2n}{\gcd(w+1, 2n)} \\
&= \gcd(w-1, 2n)q' + \gcd(v-1, n) \\
&= \gcd(w-1, 2n)q' + \gcd(w-1, n).
\end{aligned}$$

It is not possible that $\gcd(w-1, 2n) = \gcd(w-1, n)$, for the equation above would imply that $\gcd(w-1, 2n) \mid r$, again an impossibility because of Theorem 7.1.

The only remaining valid possibility is

$$\gcd(w-1, 2n) = 2\gcd(w-1, n) = 2\gcd(v-1, n),$$

which implies immediately that $\gcd(\frac{w-1}{2}, n) = \gcd(v-1, n)$ and

$$\gcd(v-1, n) \mid \left(\frac{w-1}{2}\right).$$

Therefore $T^{u+\frac{w-1}{2}} \cdot v$ is a quasipolarity, since $\frac{w-1}{2} = m\gcd(v-1, n)$ for some integer

m and

$$u + \frac{w-1}{2} = \gcd(v-1, n)(q+m) + \frac{n}{\gcd(v+1, n)},$$

a valid affine part for a quasipolarity according to Theorem 7.1. \square

Theorem 8.1. *Let n be an even number. If there exists a strong dichotomy (X_n/Y_n) with polarity $p_n = T^u \cdot v$ and the square*

$$\begin{array}{ccc} \mathbb{Z}_n & \xrightarrow{2} & \mathbb{Z}_{2n} \\ p_n \downarrow & & \downarrow p_{2n} \\ \mathbb{Z}_n & \xrightarrow{2} & \mathbb{Z}_{2n} \end{array}$$

commutes with $p_{2n} = T^{2u} \cdot w$, then there exists a strong dichotomy (X_{2n}/Y_{2n}) with polarity p_{2n} such that 2 is a morphism of dichotomies, i.e., the square

$$\begin{array}{ccc} X_n & \xrightarrow{2} & X_{2n} \\ p_n \downarrow & & \downarrow p_{2n} \\ Y_n & \xrightarrow{2} & Y_{2n} \end{array}$$

commutes.

Proof. By the two previous lemmas we know there exists an autocomplementary dichotomy V in \mathbb{Z}_n with autocomplementary function $p'_n = T^{(w-1)/2} \cdot p_n = T^{u+(w-1)/2} \cdot v$ that does not belong to the orbit of X_n nor is invariant under the antipodal map $T^{\frac{n}{2}}$. The commutativity hypothesis implies that

$$p_{2n}(2V+1) = p_{2n}(2V) + w = 2p_n(V) + w = 2p'_n(V) + 1.$$

Define $X_{2n} = (2X_n) \cup (2V+1)$ and note that $p_{2n}(X_{2n}) = X_{2n}^c$. Suppose $T^{2b} \cdot s$ is a non-trivial symmetry of X_{2n} . If $2q \in 2X_n$, then

$$T^{2b} \cdot s(2q) = 2sq + 2b = 2(sq + b) \in 2X_n$$

implying $T^b \cdot s$ is a symmetry of X_n . Hence $s = 1 + n$ and $b = n$. Thus $2b = 0$, and for any $2q+1 \in 2V+1$ we have

$$\begin{aligned} T^{2b} \cdot s(2q+1) &= T^0 \cdot (1+n)(2q+1) \\ &= 2q + 2qn + 1 + n \equiv 2 \pmod{(q+1+n)2n}. \end{aligned}$$

Therefore $2q+1+n \in 2V+1$ and $q + \frac{n}{2} \in V$. This cannot be, for V is not invariant under the antipodal map.

The remaining symmetries are of type $T^{2b+1} \cdot s$. Then for $2q+1 \in 2V+1$

$$\begin{aligned} T^{2b+1} \cdot s(2q+1) &= 2sq + s + 2b + 1 \\ &= 2\left(sq + b + \frac{s+1}{2}\right) \in 2X_n \end{aligned}$$

which is also impossible, for then $T^{b+\frac{s+1}{2}} \cdot s(V) = X_n$, and this contradicts that V is not in the orbit of X_n . □

Theorem 8.2. *There exists an infinite sequence of strong dichotomies $\{\Delta_{2^n \cdot 3}\}_{n=1}^\infty$ (with $\Delta_{2^n \cdot 3}$ in $\mathbb{Z}_{2^n \cdot 3}$) and canonical injections*

$$\Delta_6 \rightsquigarrow \Delta_{12} \rightsquigarrow \Delta_{24} \rightsquigarrow \cdots \Delta_{2^n \cdot 3} \rightsquigarrow \cdots \tag{8.1}$$

with respective polarities

$$p_n = T^{2^{n-1}} \cdot (4^{\lfloor \frac{n}{2} \rfloor} + 1). \tag{8.2}$$

In particular, Δ_{12} can be chosen in the orbit of

$$(\{0, 3, 4, 7, 8, 9\} / \{1, 2, 5, 6, 10, 11\}),$$

the classical consonance-dissonance dichotomy of Palestrina-Fux.

Proof. It is a matter of simple calculations to verify that the claimed sequence can begin with

$$\{0, 2, 3\} \rightsquigarrow \{0, 1, 4, 5, 6, 9\}$$

and be continued by induction using Proposition 8.2 and Theorem 8.1. □

Definition 8.1. *A counterpoint tower is a diagram $\Gamma : \mathbb{N} \rightarrow \mathfrak{D}\mathfrak{F}$ such that $\Gamma(j < j + 1) : \Gamma(j) \rightsquigarrow \Gamma(j + 1)$ is a monomorphism.*

In terms of this definition, Theorem 8.2 states that there exists a tower of counterpoint that includes a dichotomy in the class of the (K/D) dichotomy. Its limit exists, and it is a dichotomy in the limit of $\mathbb{Z}_{2^n \cdot 3}$.

8.3 Dense Consonances and Dissonances

Now we exhibit a counterpoint tower such that its limit is dense in S^1 . The strategy is to construct an appropriate non-rigid autocomplementary dichotomy that completes the strong dichotomy in each “floor” of the tower, as the Theorem 8.1 requires.

Lemma 8.3. *Let $A_n = \{0, \dots, 2^{n-1} - 1\}$ and $B_n = \{2^{n-1}, \dots, 2^n - 1\}$ be marked dichotomies in ambient space \mathbb{Z}_{2^n} and*

$$g = T^{2^{n-1}-1} \cdot (2^n - 1) \in \overrightarrow{\text{GL}}(\mathbb{Z}_{2^n}).$$

Then the squares

$$\begin{array}{ccc}
 A_n & \xrightarrow{3} & 3A_n \\
 g \downarrow & & \downarrow T^{3 \cdot (2^{n-1}-1) \cdot (2^n-1)} =: g' \\
 A_n & \xrightarrow{3} & 3A_n
 \end{array}$$

and

$$\begin{array}{ccc}
 B_n & \xrightarrow{T^2 \cdot 3} & 3B_n + 2 \\
 g \downarrow & & \downarrow T^{2^{n-1}+1 \cdot (2^{n+1}-1)} =: h'' \\
 B_n & \xrightarrow{T^2 \cdot 3} & 3B_n + 2
 \end{array}$$

(where $3A_n$ and $3B_n + 2$ are in $\mathbb{Z}_{2^{n \cdot 3}}$) commute. Since $gA_n = A_n$ and $gB_n = B_n$, then

$$h'(3A_n) = 3gA_n = 3A_n \quad (8.3)$$

$$h''(3B_n + 2) = 3g(B_n) + 2 = 3B_n + 2. \quad (8.4)$$

Proof. The commutativity of the first square is obvious. For the second, it is clear that

$$3(2^n - 1) \equiv 3(2^{n+1} - 1) \pmod{2^n \cdot 3},$$

while

$$\begin{aligned}
 3(2^{n-1} - 1) + 2 &= 3 \cdot 2^{n-1} - 3 + 2 \\
 &= 3 \cdot 2^{n-1} - 1 \\
 &\equiv 3 \cdot 2^{n+1} + 3 \cdot 2^{n-1} - 1 \pmod{2^n \cdot 3} \\
 &= 2^{n+2} + 2^{n-1} + 2^{n+1} + 2^n - 1 \\
 &= 2^{n+2} + 2^{n-1} + 3 \cdot 2^n - 1 \\
 &= 2^{n+2} + 2^{n-1} - 1 \\
 &= 2^{n+2} - 2 + 2^{n-1} + 1 \\
 &= (2^{n+1} - 1) \cdot 2 + (2^{n-1} + 1),
 \end{aligned}$$

and the result follows. \square

Lemma 8.4. Suppose (X/Y) is a dichotomy in $\mathbb{Z}_{2^{n-1 \cdot 3}}$ ($n \geq 2$) with polarity

$$q = T^{2^{n-2} + 2^{2^{\frac{n}{2}} - 1}} \cdot (4^{\lceil \frac{n}{2} \rceil} + 1).$$

Then the following square commutes:

$$\begin{array}{ccc} X & \xrightarrow{T^{1,2}} & 2X + 1 \\ q \downarrow & & \downarrow p_n \\ Y & \xrightarrow{T^{1,2}} & 2Y + 1, \end{array}$$

where $2X + 1$ and $2Y + 1$ are in $\mathbb{Z}_{2^n \cdot 3}$ and p_n is given by (8.2).

Proof. Note that $4^{\lceil \frac{n}{2} \rceil} + 1 \in \text{GL}(\mathbb{Z}_{2^{n-1} \cdot 3})$ since

$$4^{\lceil \frac{n}{2} \rceil} + 1 \pmod 3 = 2 \quad \text{and} \quad 4^{\lceil \frac{n}{2} \rceil} + 1 \pmod 2 = 1$$

therefore it is coprime with $2^{n-1} \cdot 3$. The commutativity follows from

$$\begin{aligned} 2(2^{n-2} + 2^{2^{\lceil \frac{n}{2} \rceil - 1}}) + 1 &= 2^{n-1} + 2^{2^{\lceil \frac{n}{2} \rceil}} + 1 \\ &= 2^{n-1} + (4^{\lceil \frac{n}{2} \rceil} + 1) \cdot 1. \end{aligned}$$

□

Now it is easy to verify that

$$(4^{\lceil \frac{n}{2} \rceil} + 1) \pmod 3 = 2$$

and

$$2^{n-2} + 2^{2^{\lceil \frac{n}{2} \rceil - 1}} \pmod 3 = \begin{cases} 0, & n \text{ even,} \\ 1, & n \text{ odd.} \end{cases}$$

This means that, if n is even

$$q(x) \pmod 3 = \begin{cases} 0 & x \pmod 3 = 0, \\ 2 & x \pmod 3 = 1, \\ 1 & x \pmod 3 = 2, \end{cases}$$

and if n is odd,

$$q(x) \pmod 3 = \begin{cases} 1 & x \pmod 3 = 0, \\ 0 & x \pmod 3 = 1, \\ 2 & x \pmod 3 = 2. \end{cases}$$

Finally, observe that h' and h'' are the identity modulo 3 for n odd and even, respectively, so using (8.3) and (8.4) we see that they extend to a non-trivial automorphism¹ of the dichotomy

¹ The inverse of its linear part is itself, since

$$(2^s - 1)^2 = 2^{2s} - 2^{s+1} + 1 \equiv 1 \pmod{2^n \cdot 3}$$

when $s \geq n$ and s is odd, for $2^n \cdot 3$ divides $2^{2s} - 2^{s+1}$.

$$U'_{n+1} = \begin{cases} \{1 + 3k\}_{k=0}^{2^n-1} \cup 3A_{n-1}, & n \text{ odd,} \\ \{1 + 3k\}_{k=0}^{2^n-1} \cup (3B_{n-1} + 2), & n \text{ even,} \end{cases}$$

with polarity q . Note that U'_{n+1} is not invariant under the antipodal map, since for n odd, we have $0 \in 3A_{n-1}$ and

$$T^{3 \cdot 2^{n-1}}(0) = 0 + 3 \cdot 2^{n-1} \notin 3A_{n-1},$$

while for n even, we have $3 \cdot (2^n - 1) + 2 \in 3B_{n-1} + 2$ and

$$T^{3 \cdot 2^{n-1}}(3 \cdot (2^n - 1) + 2) = -3 + 2 + 3 \cdot 2^{n-1} = 3 \cdot (2^{n-1} - 1) + 2 \notin 3B_{n-1} + 2.$$

Now, if X_n is a strong dichotomy in $\mathbb{Z}_{2^n \cdot 3}$ like in the proof of Theorem 8.2, then U'_{n+1} is not in its orbit, for it is not strong. Thus we know that

$$X_{n+1} = (2X_n) \cup (2U'_{n+1} + 1)$$

is a strong dichotomy in $\mathbb{Z}_{2^{n+1} \cdot 3}$ that extends X_n canonically and equivariantly. So constructed, X_{n+1} has the property

$$\{6k + 3\}_{k=0}^{2^n-1} \subset X_{n+1}, n \geq 2.$$

The limit of the counterpoint tower (8.1) can be seen as a dichotomy within the subset

$$\bigcup_{n=1}^{\infty} \left\{ \exp\left(2\pi i \frac{k}{2^n \cdot 3}\right) \right\}_{k=0}^{2^n \cdot 3-1} \subset S^1$$

which is dense in S^1 in the standard metric topology. The limit of consonances $\lim X_n$ contain the set $\{\exp(2\pi i(6k + 3)/(2^n \cdot 3)) : n, m \in \mathbb{N}\}$, which is also dense in S^1 , so $\lim X_n$ is also dense. By reducing modulo 6 all the polarities, we see that the dissonances $\lim \mathbb{Z}_n \setminus \lim X_n$ contain the set $\{\exp(2\pi i(6k + 1)/(2^n \cdot 3)) : n, m \in \mathbb{N}\}$, which is also dense.

Theorem 8.3. *There exists a counterpoint tower such that the injective limit of both its consonances and dissonances is dense in S^1 . Consequently, the interior of such limit consonances and dissonances is empty.*

Finally we show a counterpoint tower such that its limit has a continuous polarity. Consider the following facts:

1. There are 15 classes of strong dichotomies in \mathbb{Z}_{24} with polarity $T^{12} \cdot 1$.
2. For n even, the square

$$\begin{array}{ccc} \mathbb{Z}_n & \xrightarrow{2} & \mathbb{Z}_{2n} \\ T^{\frac{n}{2}} \cdot 1 \downarrow & & \downarrow T^{n \cdot 1} \\ \mathbb{Z}_n & \xrightarrow{2} & \mathbb{Z}_{2n} \end{array}$$

commutes, where $T^{\frac{n}{2}} \cdot 1$ and $T^n \cdot 1$ are quasipolarities in their respective spaces.

By Theorem 8.1 it follows from the above (by induction) that there exists a counterpoint tower

$$X_{24} \rightsquigarrow X_{48} \rightsquigarrow X_{96} \rightsquigarrow \dots$$

In the proof of Theorem 8.1, the strong dichotomies $X_{24 \cdot 2^n}$ can be constructed using the dichotomy

$$U_{24 \cdot 2^n} := \begin{cases} \mathbb{Z}_{24 \cdot 2^n} \setminus \{0, 1, 2, \dots, 12 \cdot 2^n - 1\}, & n \text{ odd,} \\ \{0, 1, 2, \dots, 12 \cdot 2^n - 1\}, & n \text{ even,} \end{cases}$$

in $\mathbb{Z}_{24 \cdot 2^n}$, since it is not strong (the symmetry $T^{12 \cdot 2^{n-1}} \cdot -1 \in \overrightarrow{\text{GL}}(\mathbb{Z}_{24 \cdot 2^n})$ is a non-trivial symmetry of $U_{24 \cdot 2^n}$) and it is not invariant under the antipodal map (it is precisely its autocomplementary function!). Thus

$$X_{24 \cdot 2^n} = 2X_{24 \cdot 2^{n-1}} \cup (2U_{24 \cdot 2^{n-1}} + 1),$$

for $n \geq 1$ is a strong dichotomy with polarity $T^{24 \cdot 2^{n-1}}$.

Example 8.1. Take the strong dichotomy

$$X_{24} = \{0, 1, 3, 5, 6, 7, 8, 9, 10, 11, 14, 16\}$$

in \mathbb{Z}_{24} . Now

$$U_{24} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$$

and

$$\begin{aligned} X_{48} = 2X_{24} \cup (2U_{24} + 1) = \{0, 1, 2, 3, 5, 6, 7, 9, \\ 10, 11, 12, 13, 14, 15, 16, 17, \\ 18, 19, 20, 21, 22, 23, 28, 32\} \end{aligned}$$

is a strong dichotomy in \mathbb{Z}_{48} .

Now the limit $X = \lim_{n \rightarrow \infty} X_{24 \cdot 2^n}$ contains the subsets

$$A = \left\{ \exp\left(2\pi i \frac{2\ell + 1}{24 \cdot 2^{2m+1}}\right) \right\}_{2\ell+1 > 12 \cdot 2^{2m+1}}$$

and

$$B = \left\{ \exp\left(2\pi i \frac{2\ell + 1}{24 \cdot 2^{2m}}\right) \right\}_{2\ell+1 < 12 \cdot 2^{2m}},$$

so X contains $A \cup B$, which is a dense subset of S^1 .

Let $p = \lim_{n \rightarrow \infty} p_{24 \cdot 2^n}$. Note that

$$\begin{aligned} \exp\left(2\pi i \frac{s}{24 \cdot 2^n}\right) &\xrightarrow{p} \exp\left(2\pi i \frac{s + 24 \cdot 2^{n-1}}{24 \cdot 2^n}\right) \\ &= \exp\left(2\pi i \left(\frac{s}{24 \cdot 2^n} + \frac{1}{2}\right)\right). \end{aligned}$$

Therefore p maps a convergent sequence of consonances $\{x_j\}_{j=1}^\infty \subseteq X \subseteq S^1$ to the convergent sequence of dissonances $\{x_j T^{i\pi}\}_{j=1}^\infty \subseteq pX \subseteq S^1$. This means that the limit extends to the continuous function

$$\begin{aligned} P : S^1 &\longrightarrow S^1, \\ x &\longmapsto xT^{i\pi}. \end{aligned}$$

Chapter 9

A Categorical Look at Gesture Theory

In this chapter and the following one, we shall describe contrapuntal rules in terms of singular homology of gestures. Chapter 9 describes the mathematical theory of gestures, whereas Chapter 10 develops singular homology for gestures and applies it to restate contrapuntal rules in terms of homological dimension.

9.1 Gestures over Topological Categories

In this section we set up the framework for a gesture theory that is based on topological categories, generalizing the case of topological spaces.¹ Let us make this precise: A category C is thought of as being a collection of morphisms, together with two maps $d, c : C \rightarrow C$ (d for “domain,” c for “codomain”), and we write $f : d(f) \rightarrow c(f)$ to make these maps evident. In what follows, we shall start from a given topological category K . This means that the collection of morphisms K is a topological space, and that the domain and codomain, as well as the composition of morphisms (on the morphism sets with the relative topologies), are continuous.

Here are two basic examples of such categories: (1) The *simplex category* ∇ associated with the unit interval I : Its morphism set is $\nabla = \{(x,y) | x,y \in I \text{ and } x \leq y\}$, $d(x,y) = (x,x)$, $c(x,y) = (y,y)$, the composition of morphisms is obvious, and the topology on ∇ is the relative topology inherited from the usual product topology on $I \times I \subset \mathbb{R} \times \mathbb{R}$. (2) The *graphical category* associated with any topological space X : Its morphism set is $X \times X$, equipped with the product topology, while we set $d(x,y) = (x,x)$, $c(x,y) = (y,y)$, and again, the composition of morphisms is obvious. If no confusion is likely, we denote the graphical category of X by X . Clearly, a graphical category is a groupoid.² In particular, the simplex category ∇ is just the subcategory of the graphical category I on the pairs (x,y) , $x \leq y$.

¹ See mathematical appendix, Section A.5

² See mathematical appendix, Section A.6.

If K, L are two topological categories, a topological functor $F : K \rightarrow L$ is a functor that is also continuous as a map between morphism sets. This defines the category \mathcal{TC} of topological categories. In order to distinguish the set of topological functors $F : K \rightarrow L$ from the larger set $K@L$ of functors, we write $K\circ L$. If X, Y are topological spaces, then the map which associates with a continuous map $f : X \rightarrow Y$ the synonymous continuous functor is fully faithful, so the category of topological spaces is a full subcategory of the category of continuous categories. Therefore, we shall henceforth tacitly identify the category **Top** of topological spaces and continuous maps with the associated category of topological categories and continuous functors.

With this in mind, if K is a topological category, the set of continuous curves with values in K is by definition the set $\nabla\circ K$. Clearly, if K is a topological space, then $\nabla\circ K \xrightarrow{\sim} I@K$, the set of continuous curves $c : I \rightarrow K$, the bijection being induced by the restriction of a functor $F : \nabla \rightarrow K$ to the canonical diagonal embedding $I \rightarrow \nabla$ of the objects in ∇ . This set is also a category if we take as morphisms between two curves $f, g : \nabla \rightarrow K$ the continuous natural transformations $\nu : f \rightarrow g$, i.e., the defining maps of the natural transformations $\nu : I \rightarrow K$ are continuous. We do, however, want it to become a topological category, and this works as follows: We take the morphism set as being composed by the triples (f, g, ν) as above. The topology is defined by the following construction. The set of objects of $\nabla\circ K$ is given the compact-open topology³ induced by the topologies of ∇ and K ; the set $I@K$ of continuous natural transformations $\nu : I \rightarrow K$ is given the compact-open topology. The triples are viewed as points in the product topology on $\nabla\circ K \times \nabla\circ K \times I@K$. Clearly, this is a topological category. Also observe that in the case of a topological space K , the compact-open topology of $I@K$ coincides with the topology induced by the isomorphism $\nabla\circ K \xrightarrow{\sim} I@K$ and the compact-open topology on $\nabla\circ K$.

Example 9.1. The set $\nabla\circ K$ can also be enforced for a category K that is not given a topology a priori as follows. Take any set $C \subset \nabla@K$ of functors $F : \nabla \rightarrow K$ into an abstract category (suppose that K is small, if set theory matters) and then select the finest topology on K such that all functors of C become continuous. For this construction one writes $\nabla\circ_C K$ to indicate that K is made a topological category via C , and that this is the set of all continuous curves with respect to this topology.

9.1.1 Digraphs Associated with Topological Categories

To generate gestures in categories, the construction of a spatial digraph [59] must be extended to topological categories. To do so, one considers the two continuous *tail and head functors* $t, h : \nabla\circ K \rightarrow K$, which are defined as follows. If $\nu : f \rightarrow g$ is a natural transformation between $f, g : \nabla \rightarrow K$, then $t(\nu) = \nu(0) : f(0) \rightarrow g(0)$, and $h(\nu) = \nu(1) : f(1) \rightarrow g(1)$. These tail and head maps are not only set maps but functors. Call this diagram of topological categories and continuous functors the

³ See mathematical appendix, Section A.5.

*spatial categorical digraph*⁴ \vec{K} of K . If one forgets about the category and just retains the objects of this configuration, one calls it the (*underlying*) *spatial digraph* of K . In particular, if Γ is a digraph, then the set of morphisms $\Gamma @ \vec{K}$ is the set of digraph morphisms into the underlying spatial digraph of K . One calls then, by definition, a *gesture with skeleton Γ and body in K* a morphism of digraphs $g : \Gamma \rightarrow \vec{K}$.

When the topological category is a groupoid, an easy proposition follows that guarantees that one may reverse all arrows, i.e., the spatial categorical digraphs of topological groupoids are self-dual.

Proposition 9.1. *If K is a topological groupoid, there is a duality automorphism $?^* : \vec{K} \xrightarrow{\sim} \vec{K}^*$ onto the dual digraph \vec{K}^* (tail and head functors exchanged). It maps a curve $g : \nabla \rightarrow K$ to the inverse curve $g^* : \nabla \rightarrow K$, defined by $g^*(x, y) = g(1 - y, 1 - x)^{-1}$.*

In other words, for a topological groupoid K , the set $\Gamma @ \vec{K}$ is in bijection with its dual set $\Gamma^* @ \vec{K}^*$, and then with the set $\Gamma^* @ \vec{K}$ associated by the duality $?^*$. The gesture $g^* : \Gamma^* \rightarrow \vec{K}$ associated by this bijection with a given gesture $g : \Gamma \rightarrow \vec{K}$ is called the *dual gesture*. It reverses the arrows of the skeleton and the morphisms of the body's curves.

9.1.2 Toward Hypergestures: The Topological Category of Gestures with Body in a Topological Category

We have constructed the set $\Gamma @ \vec{K}$ of gestures with skeleton Γ and body in a topological category K . In the theory of gestures in topological spaces [59] (section 4), which are special topological categories, this set was enriched to yield a topological space, enabling the iterative construction of hypergestures—gestures of gestures. In the present categorical setup, one accordingly wants to construct a topological category from the above set. To this end, recall that the special case $\Gamma = \uparrow$ (one arrow between two different vertices) means that one is given the topological category $\uparrow @ \vec{K} \xrightarrow{\sim} \nabla \circ K$ of continuous curves $c : \nabla \rightarrow K$ (with the above mentioned compact-open topology).

The general case follows from the observation that Γ is the colimit⁵ of the following diagram \mathcal{D} of digraphs: One takes one arrow digraph $\uparrow_a = \uparrow$ for each arrow $a \in A_\Gamma$ (A_Γ is the set of arrows of Γ) and one bullet digraph $\bullet_x = \bullet$ for each vertex $x \in V_\Gamma$ (V_Γ is the set of vertices of Γ). The morphisms are the tail or head injections $\bullet_x \rightarrow \uparrow_a$ whenever $x = t(a)$ or $x = h(a)$, therefore $\Gamma \xrightarrow{\sim} \text{colim } \mathcal{D}$. Hence the set of gestures $\Gamma @ \vec{K}$ is in bijection with the limit $\lim \mathcal{D} @ \vec{K}$ of a diagram of *objects* of topological categories $\uparrow @ \vec{K} \xrightarrow{\sim} \nabla \circ K$ (for the digraph's arrows) and $\bullet @ \vec{K} \xrightarrow{\sim} K$

⁴ See mathematical appendix, Section A.2.

⁵ See the mathematical appendix, Section A.6 for the limit and colimit concepts in categories.

(for the digraph's vertices). But the maps between these objects of categories stem effectively from functors (those from the spatial categorical graph \vec{K}). Therefore the limit can be understood as one of a diagram of topological categories. This defines a topological category, denoted by $\Gamma \vec{\@} K$, whose topology is defined as the limit topology of this diagram. In this category, a morphism is the limit of natural transformations between continuous curves and morphisms between objects of K , the latter meaning the end points of the continuous curves.

Example 9.2. If the topological category K is a topological space, we recover the topological category $\Gamma \vec{\@} K$ associated with the topological space $\Gamma \vec{\@} K$ in [59].

This definition of a topological category $\Gamma \vec{\@} K$ automatically sets forth the machinery of hypergestures known from the original topological space setup. And one accordingly has an Escher theorem for topological categories of hypergestures:

Proposition 9.2. (Escher Theorem) *If Γ, Δ are digraphs and K is a topological category, then there is a canonical isomorphism of topological categories*

$$\Gamma \vec{\@} \Delta \vec{\@} K \xrightarrow{\sim} \Delta \vec{\@} \Gamma \vec{\@} K.$$

Corollary 9.1. *The action*

$$\vec{\@} : \text{Digraph} \times \mathcal{TC} \rightarrow \mathcal{TC} : (\Gamma, K) \mapsto \Gamma \vec{\@} K$$

canonically extends to an action (denoted by the same symbol)

$$\vec{\@} : [\text{Digraph}] \times \mathcal{TC} \rightarrow \mathcal{TC} : (W, K) \mapsto W \vec{\@} K$$

of the free commutative monoid $[\text{Digraph}]$, i.e., the monoid of commutative words $W = \Gamma_1 \Gamma_2 \dots \Gamma_k$ over the alphabet Digraph of digraphs (the objects only). It is defined inductively by $\Gamma_1 \Gamma_2 \dots \Gamma_k \vec{\@} K = \Gamma_1 \vec{\@} (\Gamma_2 \dots \Gamma_k \vec{\@} K)$ and⁶ $\emptyset \vec{\@} K = K$.

Given this construction of hypergestures, one defines the category of gestures with body in K , now also including the morphisms between their skeleta. It is denoted by $\text{Gesture}(K)$. Its objects are the objects (gestures) of $\Gamma \vec{\@} K$ for any digraph Γ . Given two such gestures $g : \Gamma \rightarrow K, h : \Delta \rightarrow K$, a morphism $a : g \rightarrow h$ is a pair $a = (t, \nu)$ consisting of a digraph morphism $t : \Gamma \rightarrow \Delta$ and a morphism $\nu : g \rightarrow h \circ t$ in $\Gamma \vec{\@} K$, which we also write as a diagram but with the natural transformation being denoted by a double arrow to prevent a wrong intuition of a commutative square:

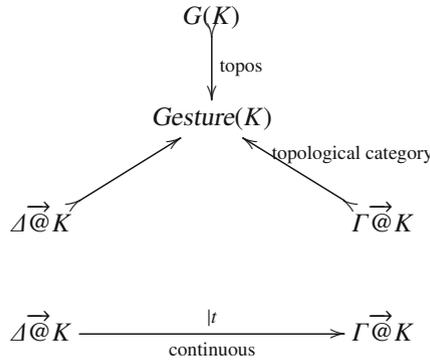
$$\begin{array}{ccc} \Gamma & \xrightarrow{g} & \vec{K} \\ \downarrow t & \searrow \text{hot} & \Downarrow \nu \\ \Delta & \xrightarrow{h} & \vec{K} \end{array}$$

⁶ \emptyset denotes the empty word.

Given a second morphism $b : h \rightarrow k$, $b = (s, \mu)$, with codomain $k : \Sigma \rightarrow \vec{K}$, the composition $b \circ a : g \rightarrow k$ is defined by $b \circ a = (s \circ t, \mu|_t \circ \nu)$, where $\mu|_t$ means that the natural transformation μ from h to $k \circ s$ is “restricted” by the digraph morphism t .

The category $Gesture(K)$ contains two types of subcategories: First, the topos⁷ $G(K) = Digraph/\vec{K} \subset Gesture(K)$ of gestures with body in K , the morphism being the digraph morphisms of gesture skeleta commuting with the domain and codomain gestures. Second, one has, for each skeleton Γ , the topological category $\Gamma@K \subset Gesture(K)$.

Finally, for each digraph morphism $t : \Gamma \rightarrow \Delta$, there is a canonical continuous restriction functor $|t : \Delta@K \rightarrow \Gamma@K$. And this is the overall picture:



9.1.3 Functoriality with Respect to the Underlying Topological Category

Given a continuous functor $F : K \rightarrow L$ between topological categories, one has a canonical morphism of spatial categorical digraphs $\vec{K} \rightarrow \vec{L}$, sending vertices to vertices by the given functor $F : K \rightarrow L$ and sending curves $f : \nabla \rightarrow K$ to curves $F \circ f$, whereas continuous natural transformations $\nu : f \rightarrow g$ are sent to the continuous natural transformations $F \circ \nu : F \circ f \rightarrow F \circ g$. This morphism of spatial categorical digraphs is called a *spatial (categorical) digraph morphism* and is denoted by \vec{F} . It canonically induces a functor $Gesture(F) : Gesture(K) \rightarrow Gesture(L)$, which is compatible with the above subcategories as made evident by the following commutative diagram:

⁷ It is a comma category of the topos of digraphs, and therefore it is a topos, see the mathematical appendix, Section A.6.

$$\begin{array}{ccc}
 G(K) & \xrightarrow{G(F)} & G(L) \\
 \downarrow & & \downarrow \\
 \text{Gesture}(K) & \xrightarrow{\text{Gesture}(F)} & \text{Gesture}(L) \\
 \uparrow & & \uparrow \\
 \Gamma @ K & \xrightarrow{\Gamma @ F} & \Gamma @ L
 \end{array}$$

The functor $\Gamma @ F$ is continuous, and the functor $G(F)$ has a number of well-known properties of functors between topoi ⁸. First, $G(F)$ is right adjoint to the base-change functor $\times \vec{F} : G(L) \rightarrow G(K)$, which associates with a gesture $g : \Gamma \rightarrow \vec{L}$ the fiber product gesture $g \times \vec{F} : \Gamma \times_{\vec{L}} \vec{K} \rightarrow \vec{K}$. Second, the base change $\times \vec{F}$ is a logical functor (it preserves all topos-theoretical constructs, such as sub-object classifiers, finite limits and colimits, and exponentials) and also has a right adjoint $\times \vec{F}_*$. Paired with this right adjoint, the base-change functor defines a geometric morphism $G(K) \rightarrow G(L)$ [51, chapter VII.1].

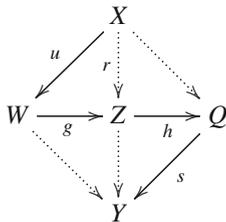
9.2 Constructing Gestures from Morphisms

To conceive a general method for generating gestures from morphisms $f : X \rightarrow Y$ in abstract categories, we make a heuristic consideration. Suppose that one is working in a musical parameter space \mathbb{R}^2 , which one endows with the structure of the Gaussian plane of complex numbers. Let us pick a rotation $e^{i\theta} : x \mapsto x \cdot e^{i\theta}$ on \mathbb{R}^2 . In linear algebra, this morphism $f = e^{i\theta}$ is a function without any relationship to gestures but acts by Fregean “teleportation” on x . This fact must be stressed, because in contradiction to algebraic reality, one imagines a rotation by angle θ by a continuous rotational movement of x around the space origin until the final position $x \cdot e^{i\theta}$ is reached. This process is visualized by the trace of x while rotating, namely by a continuous curve $c_x : I \rightarrow \mathbb{R}^2 : t \mapsto x \cdot e^{i\theta t}$ on a circle of radius $|x|$. Each intermediate position $x \cdot e^{i\theta t}$ corresponds to a factorization $f = e^{i\theta(1-t)} \circ e^{i\theta t} = f_{1-t} \circ f_t$ of f . This means that the curve $c : I \rightarrow GL_2(\mathbb{R})$ is a curve of factorizations of the given morphism f . This restatement of the gesture c in terms of factorizations means that c is interpreted as an “infinite” factorization, the factors being parametrized by the curve parameter $t \in I$.

This enables a gestural interpretation of morphisms in abstract categories. To do so, let us fix a morphism $f : X \rightarrow Y$ in a category \mathcal{C} . The *category [f] of factorizations of f* is defined as follows. Its morphism are the triples (u, g, v) of morphism $u : X \rightarrow W, g : W \rightarrow Z, v : Z \rightarrow Y$ such that $v \circ g \circ u = f$. The domain map is $d(u, g, v) = (u, Id_W, v \circ g)$, while the codomain map is $c(u, g, v) = (g \circ u, Id_Z, v)$. For

⁸ See the mathematical appendix, Section A.6.

two morphisms $(u, g, v), (r, h, s)$ with $c(u, g, v) = d(r, h, s), h : Z \rightarrow Q$, their composition is the morphism $(u, h \circ g, s)$, as shown in the following commutative diagram:



A number of evident facts follow from this construction: First, the category $[f]$ has the initial object (Id_X, Id_X, f) and the final object (f, Id_Y, Id_Y) . Second, if $k : Y \rightarrow E$ and $l : A \rightarrow X$ are morphisms, then there are two functors $[k \circ] : [f] \rightarrow [k \circ f]$ and $[\circ l] : [f] \rightarrow [f \circ l]$, respectively, sending (u, g, v) to $(u, g, k \circ v)$ and to $(u \circ l, g, v)$, respectively (keeping the above notations). Third, if C is a topological category, then so is $[f]$, if it is viewed as a subset of C^3 . Fourth, the two functors $k \circ, \circ l$ are continuous.

For any two objects X, Y in C , build the disjoint sum $[X, Y] = \coprod_{f \in X @ Y} [f]$ of the factorization categories $[f]$ (including the coproduct of topologies on the $[f]$). Then, $\nabla @ [X, Y] = \coprod_{f \in X @ Y} \nabla @ [f]$, and if $[X, Y]$ is endowed with the coproduct topology, also $\nabla \odot [X, Y] = \coprod_{f \in X @ Y} \nabla \odot [f]$. Our construction of functors from morphisms also works in this coproduct situation, and also *mutatis mutandis* for topologies on these categories. Conserving the above notations, there are two continuous functors $[k \circ] : [X, Y] \rightarrow [X, E]$ and $[\circ l] : [X, Y] \rightarrow [A, Y]$ and their associated curve functors $\nabla \odot [k \circ] : \nabla \odot [X, Y] \rightarrow \nabla \odot [X, E]$ and $\nabla \odot [\circ l] : \nabla \odot [X, Y] \rightarrow \nabla \odot [A, Y]$.

Example 9.3. If $f = Id_X$, then $[f]$ is the category of sections and retractions of X , because its objects are the triples (u, Id, v) such that $v \circ u = Id_X$.

Example 9.4. The category C is defined by a topological group G , i.e., as a category, G has one single object and the group elements as morphisms, then $[f] \xrightarrow{\sim} G$, where G is the graphical category of the topological space G . In fact, the morphisms of $[f]$ are the triples (u, g, v) of elements of G such that $v \circ g \circ u = f$. Since any two of them are free and determine the third, we take the morphisms as being the pairs $(d, c) \in G \times G$, where we have $u = d, g = c \circ u^{-1}, v = f \circ u^{-1} \circ g^{-1}$. The topology is the product topology of $G \times G$.

For example, if C is defined by the cartesian product group $G = \mathbb{R}^n \times \overrightarrow{GL}_n(\mathbb{R})$ of the additive group \mathbb{R}^n and the general affine group $\overrightarrow{GL}_n(\mathbb{R})$, $[f] \xrightarrow{\sim} \mathbb{R}^n \times \overrightarrow{GL}_n(\mathbb{R})$, the topological space category of pairs (x, g) of points x in \mathbb{R}^n and affine transformations $g : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$. We then have a continuous (group action) functor $\epsilon : \mathbb{R}^n \times \overrightarrow{GL}_n(\mathbb{R}) \rightarrow \mathbb{R}^n : (x, g) \mapsto g(x)$ into the topological category \mathbb{R}^n deduced from the group \mathbb{R}^n . Therefore, from a gesture $g : \Gamma \rightarrow \mathbb{R}^n \times \overrightarrow{GL}_n(\mathbb{R})$, we deduce a gesture $\epsilon \circ g : \Gamma \rightarrow \mathbb{R}^n$. The latter is a gesture whose curves are just continuous curves in real n -space, but they are not arbitrary, since they are induced by curves of points and linear transformations. This very special case reveals the power of our construction of factor

categories: They include the concept of gestures of transformations of points, not merely abstract topological gestures. A gesture in $\mathbb{R}^n \times \overrightarrow{GL}_n(\mathbb{R})$ might specify curves as well specify curves that are more general than just curves of transformations. Let us take a closer look at this.

As in our initial example in \mathbb{R}^2 of a rotational curve $c(t) = (x, e^{i\theta t})$, we can vary the transformation and fix the point x , but we might also just take an arbitrary continuous curve $d(t) = (x(t), Id_{\mathbb{R}^2})$ in \mathbb{R}^2 and let the transformation remain the identity. More generally, we may vary both, the point and the curve, and consider a curve $e(t) = (x(t), g(t))$ in $\mathbb{R}^2 \times \overrightarrow{GL}_2(\mathbb{R})$. This opens the concept of a gesture whose curves are characteristic in that they may pertain to transformational constructs, to purely topological rationales, or to both. Such a setup works for any (topological) group action on a given module, such as the musically relevant action of the general affine group $\overrightarrow{GL}(\mathbb{Z}_{12})$ on the pitch common class group \mathbb{Z}_{12} (with the discrete topology, for example).

9.2.1 Interpreting Diagrams as Gestures

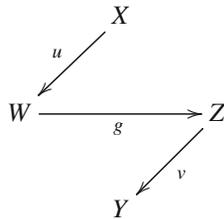
From Example 9.3 we learn that one should analyze the category of factorizations for module categories more carefully, since sections and retractions define direct summands in the abelian categories. With this in mind, we first construct certain standard gestures. To begin with, let $g : W \rightarrow Z$ be any morphism in a category C . There is a functor $\searrow(g) : \nabla \rightarrow C$ defined by

$$\searrow(g)(x, y) = \begin{cases} Id_W & \text{if } x = y = 0, \\ g & \text{if } 0 = x < y, \\ Id_Z & \text{if } 0 < x. \end{cases} \tag{9.1}$$

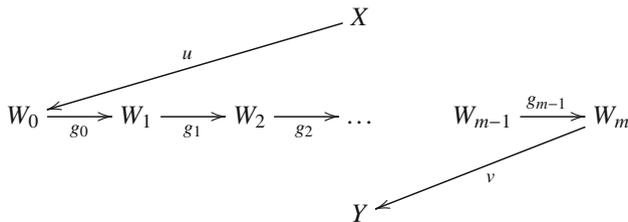
This approach enables the construction of gestures from diagrams in categories as follows. Let K be a small category (its objects define a set). Then take the topology on K such that all functors $c : \nabla \rightarrow K$ are curves, i.e., we take $\nabla @_{\nabla} K$. Consider K as a digraph with the two maps $d, c : Mor(K) \rightarrow Ob(K)$ from the morphism set $Mor(K)$ to the object set $Ob(K)$. Then we have the following morphism of digraphs $\searrow : K \rightarrow \overrightarrow{K}$, which sends a morphism $f : X \rightarrow Y$ to the curve $\searrow(f)$ with tail X and head Y . Therefore, if we have any diagram $\delta : \Delta \rightarrow K$ in the category K , we may compose it with \searrow and obtain a gesture $\searrow \circ \delta : \Delta \rightarrow \overrightarrow{K}$. It is denoted by $\overrightarrow{\delta}$, and we call it the *discrete gesture associated with the diagram* δ . This clearly extends to a *discrete gesture functor* $\overrightarrow{?} : \Delta @ K \rightarrow \Delta @ \overrightarrow{K}$ from the category of diagrams and natural transformations to the category of gestures of these spaces.

9.2.2 Gestures with Bodies in Factorization Categories

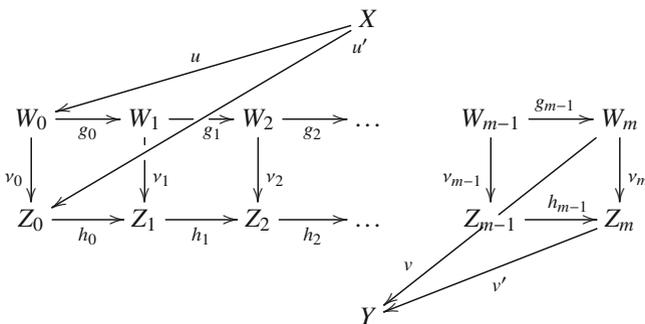
A morphism (u, g, v) in a factorization category $[f]$ with



yields a curve $\searrow (u, g, v) : \nabla \rightarrow [f]$. This construction may be iterated as follows: For any sequence $g. = (g_i : W_i \rightarrow W_{i+1})_{i=0,1,\dots,m-1}$ of length m of morphisms in C , there is a functor $\searrow (g.) : \nabla \rightarrow C$ where the restriction $\searrow (g.)|_{\nabla_{[i/m, (i+1)/m]} \rightarrow C}$ to the full subcategory $\nabla_{[i/m, (i+1)/m]} = \{(x, y) | i/m \leq x \leq y \leq (i+1)/m\}$ of ∇ is the above one-step construction for $g_i : W_i \rightarrow W_{i+1}$. We may therefore also consider curves $\searrow (u, g., v)$ associated with the chain $(u, g., v)$ of morphisms in $[f]$:



Given two such curves $\searrow (u, g., v), \searrow (u, h., v)$ of the same length m , the second one involving the morphisms $h_i : Z_i \rightarrow Z_{i+1}$ and $u' : X \rightarrow Z_0, v' : Z_m \rightarrow Y$, a morphism $v. : \searrow (u, g., v) \rightarrow \searrow (u, h., v)$ is a natural transformation consisting of a chain of morphisms $v. = (v_0 : W_0 \rightarrow Z_0, \dots, v_m : W_m \rightarrow Z_m)$ such that we have this commutative diagram:



Example 9.5. Let us give an example of the Escher theorem that has musical relevance. Let G be a topological group, X a topological space, and $G \times X \rightarrow X$ a

continuous group action. Denote by ${}^G X$ the topological category whose objects are the elements of X , and whose morphisms $g : x \rightarrow y$ are the triples $(x, y, g) \in X^2 \times G$ such that $y = gx$, the topology of ${}^G X$ being induced from the product topology on $X^2 \times G$. If the topologies are all indiscrete, a continuous curve $F : \nabla \rightarrow {}^G X$ is just a functor.

A classical example for such a topological category ${}^G X$ from transformational music theory is the canonical action of the general affine group $G = \overrightarrow{GL}(\mathbb{Z}_{12}) = T^{\mathbb{Z}_{12}} \rtimes \mathbb{Z}_{12}^\times$, on the pitch class set $X = \mathbb{Z}_{12}$, together with the indiscrete topology.

Let us identify ${}^G X$ morphism with discrete gestures constructed in Section 9.2.1. Composing such discrete gestures, we may more generally define discrete gestures $\searrow (g_1, g_2, \dots, g_n)$ for any sequence g_1, g_2, \dots, g_n of morphisms that can be composed to $g_1 \circ g_2 \circ \dots \circ g_n$, see Section 9.2.2. Let \uparrow be the digraph with two vertices t (tail), h (head) and one connecting arrow a from t to h . Then we can see that $\searrow (g_1, g_2, \dots, g_n) \in \uparrow \overset{\rightarrow}{@} {}^G X$.

In our musical example, we may interpret a consonant interval as being a discrete gesture

$$\searrow (g) \in \uparrow \overset{\rightarrow}{@} \overrightarrow{GL}(\mathbb{Z}_{12})\mathbb{Z}_{12}$$

associated with a morphism $g : c \rightarrow d$ from a cantus firmus pitch class c to discantus pitch class d , and where $g = T^k$ is the translation by a Fux consonance $k = d - c \in K = \{0, 3, 4, 7, 8, 9\}$. If $\searrow (g_1), \searrow (g_2), \dots, \searrow (g_n)$ is a sequence of n such discrete interval gestures, stemming from morphisms $g_i : c_i \rightarrow d_i, i = 1, \dots, n$, we have the morphisms $f_i : \searrow (g_i) \rightarrow \searrow (g_{i+1}), i = 1, \dots, n-1$ defined by the natural transformations on curve arguments 0, 1 via $f_i(0) = T^{c_{i+1}-c_i}, f_i(1) = T^{d_{i+1}-d_i}, i = 1, \dots, n-1$. This defines a hypergestural curve

$$\searrow (f_1, f_2, \dots, f_n) \in \uparrow \uparrow \overset{\rightarrow}{@} \overrightarrow{GL}(\mathbb{Z}_{12})\mathbb{Z}_{12},$$

which formally represents a first-species contrapuntal sequence of n consonant intervals. Applying the Escher theorem for the exchange permutation of the first and the second digraph \uparrow , the sequence $\searrow (f_1, f_2, \dots, f_n)$ corresponds to the discrete curve from the cantus firmus curve $\searrow (c_{1,2}, c_{2,3}, \dots, c_{n-1,n})$ defined by the translational morphisms $c_{i,i+1} : c_i \rightarrow c_{i+1}$ to the discantus curve $\searrow (d_{1,2}, d_{2,3}, \dots, d_{n-1,n})$ defined by the translational morphisms $d_{i,i+1} : d_i \rightarrow d_{i+1}$ for $i = 1, 2, \dots, n-1$. These two interpretations of a contrapuntal sequence correspond to the two musicological interpretations of the word “contra”. The sequence $\searrow (f_1, f_2, \dots, f_n)$ is the original meaning, that the opposition is between successive intervals. The other interpretation, that the discantus curve is the “opposition” to the cantus firmus line, is well known but historically not adequate, as discussed in Section 1.3. The Escher theorem mediates between these two interpretations.

9.2.3 Homological Extensions Are Gestures

Let us look at the special case where $C = {}_R\mathbf{Mod}$, the category of left R -modules and linear homomorphisms over a commutative ring R , and let us take the factorization category $[0]$ of the zero homomorphism $0 : 0 \rightarrow 0$ on the zero module. Consider now two exact sequences, one g , of modules W , and one h , of modules Z , and generate their associated curves as defined above; we call them *exact curves*. Then the above morphism v is just a morphism between exact sequences, meaning that the category of exact sequences is a canonical subcategory of the category of curves in $[0]$. If we look in particular at short exact sequences (length 2), and we restrict to morphisms between sequences of common initial module W and terminal module Z , we get the groupoid of exact sequences, and the isomorphism classes define the classical set $Ext_R(Z, W)$ of congruence classes of extension of Z by W [50]. We therefore have this fact:

Fact 9.1. *The categories of factorization are a natural extension of structures from homological algebra encountered, for example, in the construction of $Ext_R^n(Z, W)$.*

Chapter 10

Hypergesture Homology for Counterpoint

In this chapter, we develop gestural singular homology and restate the condition of maximal intersection (see Definition 2.5 or Section 10.2) in terms of a dimension of a first homology module.

10.1 Singular Homology for Hypergestures

A homological approach to gestures is based on the following observations. Singular homology starts with continuous functions on standard objects, either n -dimensional simplexes or n -dimensional cubes.¹ It is well known that both, the simplicial and the cubical homology, yield the same homology groups. Our approach is based on cubic homology. This one considers continuous functions $s : I^n \rightarrow X$ on the n -dimensional cubes, n -fold cartesian products $I^n = I \times I \times \dots \times I$ of the real unit interval I , with values in a topological space X . These functions are called *singular n -cubes*.

Example 10.1. Some singular cubes on the torus surface $X = \mathbb{T}^2$ are shown in [Figure 10.1](#). A singular 0-cube is a map σ_0 from the singleton I^0 to \mathbb{T}^2 , a singular 1-cube is a continuous line map $\sigma_1 : I \rightarrow \mathbb{T}^2$, and a 2-cube is a continuous square surface map $\sigma_2 : I^2 \rightarrow \mathbb{T}^2$.

For a given (unitary) commutative ring² R , homology is defined starting from the modules $C_n(R, X)$ of n -chains, i.e., the formal R -linear combination of singular n -cubes. The “yoga” of homology is that one can map n -chains to $n - 1$ -chains by the R -linear boundary homomorphism $\partial_n : C_n(R, X) \rightarrow C_{n-1}(R, X)$ such that $\partial_n \circ \partial_{n+1} = 0$. This means that $B_{n+1} = \text{Im}(\partial_{n+1}) \subset Z_n = \text{Ker}(\partial_n)$. The quotients $H_n(R, X) = Z_n/B_{n+1}$ are called *n th homology modules*. One of their signification is that their dimension may measure holes in the topological space X .

¹ See the mathematical appendix, Section A.5.

² See the mathematical appendix, Section A.4.

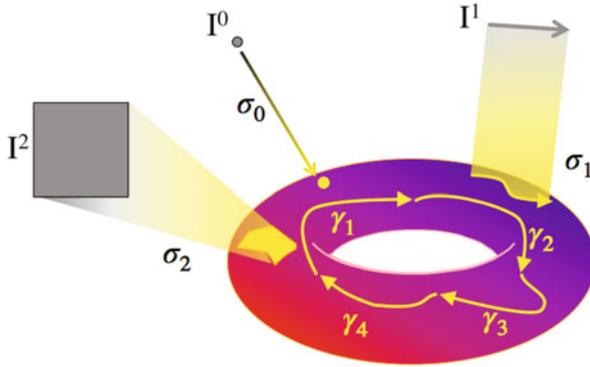


Fig. 10.1 Singular cubes on the torus. A 0-cube is a point, a 1-cube is a continuous line map and a 2-cube is a continuous square surface map. And four 1-cubes circumscribe the torus hole.

Example 10.2. Let us make a simple example for the torus. If we look at a singular 1-cube σ_1 , its boundary $\partial_1\sigma_1$ is an alternate sum of its faces (the two curve endpoints): $\partial_1\sigma_1 = \sigma_1|_{x=1} - \sigma_1|_{x=0}$, where x is the curve parameter on I . If we look at a singular 2-cube σ_2 , its boundary $\partial_2\sigma_2$ is an alternate sum of its four faces $\sigma_2|_{x=0}, \sigma_2|_{x=1}, \sigma_2|_{y=0}, \sigma_2|_{y=1}$, which are the restrictions of σ_2 to its four “face” subsets defined for the conditions $x = 0, x = 1, y = 0, y = 1$ on the I^2 ’s coordinates x and y . These four contiguous singular 1-cubes define a 1-chain, namely the alternate sign sum $\sigma_2|_{x=1} - \sigma_2|_{x=0} + \sigma_2|_{y=0} - \sigma_2|_{y=1}$. But it is clear that if we concatenate four contiguous singular 1-cubes $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ as shown in [Figure 10.1](#), their sum cannot stem from the boundary of a singular 2-cube, since the torus has a hole that prevents the existence of such a surface on \mathbb{T}^2 . This means that the hole circumscribed by the chain $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ in Z_1 defines a non-zero element of $H_1(\mathbb{R}, \mathbb{T}^2)$.

Homology and gesture theory are connected as follows: We have $\mathbf{Top}^@ (I^n, X) \xrightarrow{\sim} \mathbf{Top}^@ (I, \mathbf{Top}^@ (I, \dots \mathbf{Top}^@ (I, X) \dots))$. Because $\mathbf{Top}^@ (I, X) \xrightarrow{\sim} \uparrow @ \vec{X}$, $\mathbf{Top}^@ (I^n, X)$ can be identified with the space $\uparrow @ \uparrow @ \dots \uparrow @ \vec{X}$ of n -fold hypergestures in X for the sequence $\uparrow, \dots \uparrow$ of n copies of the one-arrow digraph \uparrow . This means that singular n -cubes in X are very special hypergestures.

In other words, the basic objects of (cubic) homology are special hypergestures. Therefore, it is logical to consider the task of defining homology theory for hypergestures, and we shall do so for the categorical hypergestures. *We shall introduce homology based upon a sequence $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ of arbitrary digraphs instead of the classical sequence of one-arrow digraphs \uparrow .* We had shown in Section 9.2.1 that homological extensions are also special concepts derived from gestural constructions related to categories of factorizations. So the present approach is a further example of the conceptual homological power of mathematical gesture theory.

10.1.1 Chain Modules for Singular Hypergesture Homology

We shall consider a commutative ring R as a basic coefficient domain for modules in homology. We first need the modules generated by singular cubes. If $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ is a sequence of n digraphs, and if K is a topological category, a *gestural n -cube* is, by definition, a hypergesture $s \in \Gamma_n \Gamma_{n-1} \dots \Gamma_1 \xrightarrow{\circlearrowright} K$. We denote by $R\Gamma_n \Gamma_{n-1} \dots \Gamma_1 \xrightarrow{\circlearrowright} K$ the free R -module over the basis $\Gamma_n \Gamma_{n-1} \dots \Gamma_1 \xrightarrow{\circlearrowright} K$. The elements of this module are called *gestural n -chains*. The 0-chains are the elements of RK , the free R -module over the objects of K .

As we will look at n -chains for sequences of not necessarily identical digraphs, we need to specify the definition of such modules for variable digraphs. In the present context, we want to choose a type of digraph sequence that provides us with a finiteness condition, namely that our infinite sequence Γ_i should use only a finite number of digraphs. Let us denote them by $\Gamma_0, \Gamma_1, \dots, \Gamma_{d-1}$. Then the sequence Γ_i may be encoded as a d -adic number $0.c_1, c_2, \dots, c_i, c_{i+1}, \dots$, where each entry $0 \leq c_i < d$ refers to one of the digraphs of our selection $\Gamma_0, \Gamma_1, \dots, \Gamma_{d-1}$. The total information, sequence of digraphs and the basic topological category K , will be encoded by the d -adic “number” $K.c. = K.c_1, c_2, \dots, c_i, c_{i+1}, \dots$. For example, if we take the classical sequence in cubic singular homology in a topological space X of constant digraphs \uparrow , encoded by $\emptyset = \Gamma_0, \uparrow = \Gamma_1$, we have the encoding $X.1111\dots$; or for a finite theory $X.1111\dots 11000000\dots$. We could also take $\emptyset = \Gamma_0, \uparrow = \Gamma_1, t \circlearrowleft T = \Gamma_2$, the last digraph being the final digraph $\mathbf{1}$ with loop T in the category of digraphs. The code $S^1.210000\dots$ would then encode the hypergestures in $\uparrow \mathbf{1} \xrightarrow{\circlearrowright} S^1$ in the unit circle S^1 , the space $\emptyset \uparrow \mathbf{1} \xrightarrow{\circlearrowright} S^1$ being a singleton.

For a sequence $K.c.$ we can now define the n -chain modules. The module C_n of n -chains is the direct sum of all modules defined by selecting any partial sequence of n (not necessarily contiguous) digraphs in the digraph sequence. This means that we have to look at the n -length hypergestures defined by a partial sequence in the d -adic “number”, $c_{j_1}, c_{j_2}, \dots, c_{j_n}$, with $j_1 < j_2 < \dots < j_n$, on $\Gamma_{c_{j_n}}, \dots, \Gamma_{c_{j_1}} \xrightarrow{\circlearrowright} K$. We define $C_n = C_n(K.c.)$ to be the direct sum of all these length- n hypergesture R -modules $R\Gamma_{c_{j_n}}, \dots, \Gamma_{c_{j_1}} \xrightarrow{\circlearrowright} K$, with the restriction that for any such n -length sequence, we take it only once if it would occur for different index sequences. For example, in the classical case of cubic singular homology $X.1111\dots$, we have $C_n(K.1111\dots) = R \uparrow \uparrow \dots \uparrow \xrightarrow{\circlearrowright} K$, the R -module generated by the hypergestures on n copies of \uparrow with values in K . We have built the sequence of n -chain modules, starting with $C_0(K.c.) = RK$. Next, we define the boundary homomorphisms $\partial_n : C_n \rightarrow C_{n-1}$. As usual, we set $C_n = 0$ for negative n , and therefore $\partial_n = 0$ for $n \leq 0$.

10.1.2 Boundary Homomorphisms

The boundary operator in classical homology $\partial_n = \partial$ on n -chains $c = \sum_i r_i g_i$ is the R -linear extension of its action on the basis, the gestural n -cubes:

$$\partial(\sum_i r_i g_i) = \sum_i r_i \partial g_i.$$

Let us therefore select a length- n sequence $\Gamma_0, \Gamma_1, \dots, \Gamma_i, \dots, \Gamma_{n-1}$ of digraphs and look at the boundary operator on a single gestural n -cube $g \in \Gamma_0 \Gamma_1 \dots \Gamma_i \dots \Gamma_{n-1} \overrightarrow{\textcircled{K}}$. The classical formula has two components: the face operator $g \mapsto g_{i,\alpha}$, and then the alternating combination of faces:

$$\partial g = \sum_{i=0}^{n-1} \sum_{\alpha=0,1} (-1)^{i+\alpha} g_{i,\alpha}.$$

The face operator creates a chain in C_{n-1} , and this works by means of evaluation of the original singular cube on $n-1$ -dimensional subcubes of g , its faces. The classical situation makes two choices to define such faces. First, we select a dimension $i = 0, 1, \dots, n-1$ in the cube I^n . This coordinate space I has the two extremal values $\alpha = 0, \alpha = 1$. The faces in this dimension are the restrictions of the singular cube g (which is a function!) to the subsets $I^n | \alpha = 0, I^n | \alpha = 1$, which are functions on the remaining $n-1$ coordinates, in fact singular $n-1$ cubes; we denote them by $g_{i,\alpha}$.

Let us generalize this classical setup to gestures as follows. First, observe that in the gestural understanding of the unit interval, 0 is just the tail argument, and 1 is the head argument of the continuous curve given on the digraph \uparrow . In other words, if we have a singular cube $g : I \rightarrow X$, this corresponds to the gesture $g^* : \uparrow \rightarrow \overrightarrow{X}$. The evaluation at tail t and head h of \uparrow then corresponds to the restriction of g^* to the two vertices of \uparrow , yielding two points $g^*(t), g^*(h)$ of X : The classical formula suggests that we take the head point with positive sign α , and the tail point with negative sign α , yielding $g^*(h) - g^*(t)$.

This works for the case of one single arrow, but for general digraphs, we need a formula that takes care of all possible arrows (if any). We recognize that the head value is the result of calculating the gesture on the digraph that results from (1) omitting the tail vertex t and (2) taking the digraph resulting from all that is left after removing the arrows that are connected to t . Therefore, the general procedure for a digraph Γ is to (1) select an arrow $a \in A_\Gamma$ in the arrow set A_Γ of Γ , then (2) take the tail t_a of a and (3) restrict to the digraph $\Gamma|a^-$ obtained from removing t_a and all arrows connected to t_a . The analogue procedure for the head of a would yield the restricted digraph $\Gamma|a^+$, resulting from removing the head h_a as well as all arrows connected to this head. For our elementary situation, we have $\uparrow|a^- = \{h\}, \uparrow|a^+ = \{t\}$.

We next have to manage to exhaust all arrows. We may define the face of a gesture $g \in \Gamma \overrightarrow{\textcircled{K}}$ on a discrete digraph Γ as the sum $\sum_{v \in V_\Gamma} g(v)$ of the values of g on the

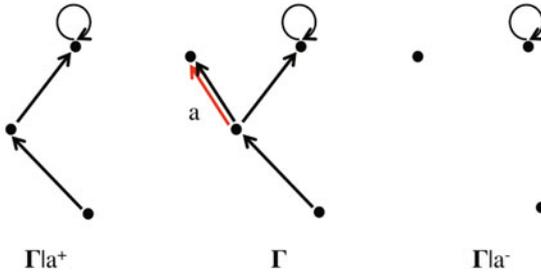


Fig. 10.2 The two reductions of the skeleton digraph of a gesture.

digraph’s vertices. The general case for these face operations can be set recursively as follows: We restrict the situation to the digraphs $\Gamma|a^-$ and $\Gamma|a^+$, see Figure 10.2.

In the initial situation of $\Gamma = \uparrow$, we have only points left. In the general case, there will be arrows left in both, $\Gamma|a^-$ and $\Gamma|a^+$ after omitting the tail and head of arrow a . But we now have fewer arrows and vertices. Therefore we can apply the recursive definition of face construction. This yields

Definition 10.1. Let $g : \Gamma \rightarrow \vec{K}$ be a gesture. Then its face g^\square is defined as follows, with values in the chain module RK :

1. If $A_\Gamma = \emptyset$ (discrete skeleton), then we set

$$g^\square = \sum_{v \in V_\Gamma} g(v).$$

2. The general case is defined recursively on the number of arrows in A_Γ :

$$g^\square = \sum_{a \in A_\Gamma} (g|a^-)^\square - (g|a^+)^\square.$$

3. The face homomorphism is canonically extended by linearization to the face homomorphism $R\Gamma \xrightarrow{\vec{\otimes}} K \rightarrow RK$ on chains $x = \sum_g c_g g$ by

$$x^\square = \sum_g c_g g^\square.$$

Up to this point, our procedure for gestures was restricted to the digraph that is the first (left) in the sequence of digraphs defining hypergestures. This corresponds to taking the faces with respect to the first coordinate of the hypercube I^n . In the definition of the boundary operator ∂ , one takes faces with respect to all the other coordinates. To deal with this more general coordinate selection, we use the Escher theorem. It allows us to transform the i th coordinate space to the first one and to perform face operators there. More precisely, if $g \in \Gamma_0 \Gamma_1 \dots \Gamma_{n-1} \xrightarrow{\vec{\otimes}} K$, and if $0 \leq i < n$,

we consider the permutation π_i that maps the i th index to the first and leaves the others unaltered, resulting in a sequence $\Gamma_i\Gamma_0\dots\Gamma_{i-1}\Gamma_{i+1}\dots\Gamma_{n-1}\xrightarrow{\text{}}K$, and then the Escher theorem yields a hypergesture $g_i \in \Gamma_i\Gamma_0\dots\Gamma_{i-1}\Gamma_{i+1}\dots\Gamma_{n-1}\xrightarrow{\text{}}K$, corresponding to g . We also extend by linearization this map $g \mapsto g_i$ to the chain modules generated by these gesture spaces. To g_i we apply the face operator, resulting in a hypergesture $g_i^\square \in \Gamma_0\dots\Gamma_{i-1}\Gamma_{i+1}\dots\Gamma_{n-1}\xrightarrow{\text{}}K$, the i th face of g . Using this construction, the boundary operator is defined by

$$\partial_n g = \sum_{i=0}^{n-1} (-1)^i g_i^\square,$$

it yields a chain in C_{n-1} , and we finally extend this operator linearly to C_n . Observe that the power $(-1)^i$ is the signature of π_i . It follows directly from the above linearization procedures for Escher correspondences and faces that for any chain $x \in C_n$, we have

$$\partial_n x = \sum_{i=0}^{n-1} (-1)^i x_i^\square.$$

We are left with the task of showing that this boundary operator verifies the crucial equation $\partial^2 = 0$, which then enables us to define the homology modules:

Proposition 10.1. *Keeping the above notations, the composition $\partial_{n-1} \circ \partial_n$ is the zero homomorphism $C_n \rightarrow C_{n-2}$.*

Proof. It suffices to show that $\partial^2 = 0$ on hypergestures $g \in \Gamma_0\Gamma_1\dots\Gamma_{n-1}\xrightarrow{\text{}}K$. If all digraphs are discrete, we have $\Gamma_0\Gamma_1\dots\Gamma_{n-1}\xrightarrow{\text{}}K \xrightarrow{\sim} K^{\Gamma_0 \times \Gamma_1 \times \dots \times \Gamma_{n-1}}$, and the vanishing of ∂^2 follows from the proof idea in the classical case of cubic homology, namely that for $i \leq j$, $g_i(x)_j(y) = g_{j+1}(y)_i(x)$, and they cancel out since the signs of the -1 coefficients are different. The same argument works for the case where all digraphs Γ_i are isomorphic to \uparrow or discrete.

For the general case, we have

$$\begin{aligned} \partial_{n-1}(\partial_n g) &= \sum_{i=0}^{n-1} (-1)^i \partial_{n-1}(g_i^\square) \\ &= \sum_{i=0}^{n-1} (-1)^i \sum_a^{A\Gamma_i} \partial_{n-1}((g_i|a^-)^\square) - \partial_{n-1}((g_i|a^+)^\square) \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-2} \sum_{(a,b)}^{A\Gamma_i \times A\Gamma_j} (-1)^{i+j} (((g_i|a^-)_j^\square|b^-)^\square - ((g_i|a^-)_j^\square|b^+)^\square) \\ &\quad - ((g_i|a^+)_j^\square|b^-)^\square + ((g_i|a^+)_j^\square|b^+)^\square \end{aligned}$$

As in the classical case, we need to show that for any sign combination $\alpha, \beta \in \{+, -\}$, and $i \leq j$, we have

$$((g_i|a^\alpha)_j|b^\beta)^\square = ((g_{j+1}|b^\beta)_i|a^\alpha)^\square.$$

The left side is

$$((g_i|a^\alpha)_j|b^\beta)^\square = \left(\sum_c^{A_{a^\alpha}} (g_i|a^\alpha|c^-)^\square - (g_i|a^\alpha|c^+)^\square \right)_j |b^\beta)^\square.$$

Because the Escher permutation in j , the face operator, and the restriction $|b^\beta$ are all linear, this yields

$$((g_i|a^\alpha)_j|b^\beta)^\square = \sum_c^{A_{a^\alpha}} \left(((g_i|a^\alpha|c^-)^\square)_j |b^\beta)^\square - (((g_i|a^\alpha|c^+)^\square)_j |b^\beta)^\square \right).$$

But the double restriction $|a^\alpha|c^\pm$ commutes, therefore $g_i|a^\alpha|c^\pm = g_i|c^\pm|a^\alpha$ and also $g_i|c^\pm = h_i$ for an Escher transformed h_i of a h defined on c^\pm instead of Γ_i in the i th coordinate. Therefore, recursion on the size of Γ_i yields

$$((h_i|a^\alpha)_j|b^\beta)^\square = ((h_{j+1}|b^\beta)_i|a^\alpha)^\square$$

for all these restricted hypergestures h . Since these restrictions cover all of a^α , this implies $((g_i|a^\alpha)_j|b^\beta)^\square = ((g_{j+1}|b^\beta)_i|a^\alpha)^\square$, and since these components appear with opposite sign, we are done. \square

Therefore $Im(\partial_{n+1}) \subset Ker(\partial_n)$, and we can define homology modules:

Definition 10.2. Keeping the above notations, for a sequence $K.c.$ of digraphs $c.$ and topological category K , we have a chain complex

$$\dots \longrightarrow C_n(K.c.) \xrightarrow{\partial_n} C_{n-1}(K.c.) \longrightarrow \dots C_0(K.c.) \longrightarrow 0$$

and we may define hypergestural homology modules

$$H_n(K.c.) = Ker(\partial_n) / Im(\partial_{n+1})$$

for all n .

The following result about homological functoriality follows directly:

Proposition 10.2. If $f : K \rightarrow L$ is a topological functor, and if we are given a sequence $\Gamma.$ of digraphs, encoded as number $c.$ as above, then the canonical morphism of chain complexes $C(f) : C(K.c.) \rightarrow C(L.c.)$ induces a canonical sequence, functorial in f , of homology module homomorphisms $H_n(f) : H_n(K.c.) \rightarrow H_n(L.c.)$.

This functoriality implies that each sequence $\Gamma.$ of digraphs generates a sequence of invariants on topological categories K that comprise the classical homology modules associated with the constant sequence $K.\bar{1}$.

Example 10.3. Let us first look at an example involving different digraphs, namely the case mentioned above, $K.210000\dots = K.21\bar{0}$, encoding the hypergestures in $\uparrow \mathbf{1}\bar{\otimes}K$ in K . We have this chain complex:

$$0 \longrightarrow R \uparrow \mathbf{1}\bar{\otimes}K \xrightarrow{\partial_2} R \uparrow \bar{\otimes}K \oplus R\mathbf{1}\bar{\otimes}K \xrightarrow{\partial_1} RK \longrightarrow 0.$$

If $g \in \uparrow \bar{\otimes}K$, we have $\partial g = g(h) - g(t)$, which in a musical situation means to express the formal difference between end state of gesture g and its initial state.

If $g \in \mathbf{1}\bar{\otimes}K$, then $\partial g = 0$. Defining $D_1 : R \uparrow \bar{\otimes}K \rightarrow RK$ by linear extension to the free module by $g \mapsto \Delta(g) = g(h) - g(t)$, we have $Im(\partial_1) = Im(D_1)$ and therefore

$$H_0 = RK/Im(D_1).$$

If R is a field, and if K is a topological space, $dim(H_0)$ is well known to be the number of pathwise components of K .

Furthermore, $Ker(\partial_1) = Ker(D_1) \oplus R\mathbf{1}\bar{\otimes}K$. Let us calculate $Im(\partial_2)$. For $g \in \uparrow \mathbf{1}\bar{\otimes}K$, we have

$$\partial_2 g = g_0^\square - g_1^\square$$

with $g_1^\square = 0$ since $a^\pm = \emptyset$, and $g_0^\square = g(t) - g(h) \in \mathbf{1}\bar{\otimes}K$. Similarly with the above, we now define $D_2 : R \uparrow \mathbf{1}\bar{\otimes}K \rightarrow R\mathbf{1}\bar{\otimes}K$ via $D_2(g) = g(h) - g(t)$, and we have $Im(\partial_2) = Im(D_2) \subset R\mathbf{1}\bar{\otimes}K$. Therefore

$$H_1 = Ker(D_1) \oplus R\mathbf{1}\bar{\otimes}K/Im(D_2),$$

where the right factor is generated by homotopy classes of loops, and the left one is generated by loops. Finally, we have $H_2 = Ker(D_2)$, the space of loops of loops.

For a first topological category, take $K = S^1$, the topological space of the unit circle. The factor $R\mathbf{1}\bar{\otimes}K/Im(D_2)$ is the fundamental group R -algebra of homotopy classes of loops, since all classes are represented by loops at a fixed point. Their group composition is also well-defined: $H_1(S^1.21\bar{0}) = Ker(D_1) \oplus R\pi_1(S^1)$.

For a second topological category, we consider the topological category defined by a topological group $K = G$, which is quite the contrary to the category defined by a topological space: We have a single object, the group's identity $Id \in G$, whereas the morphisms $x : Id \rightarrow Id$ are the group elements, and their composition $x \circ y$ is the group operation. A curve $g : \nabla \rightarrow G$ is a continuous function g with $g(y, z) \circ g(x, y) = g(x, z)$ for any $0 \leq x \leq y \leq z \leq 1$. For example, if $G = SL_2(\mathbb{R})$, then

$$\text{the shearing } s(x, y) = \begin{pmatrix} 1 & y-x \\ 0 & 1 \end{pmatrix} \text{ and the dilation } d(x, y) = \begin{pmatrix} e^{y-x} & 0 \\ 0 & e^{x-y} \end{pmatrix}$$

are examples of such curves.

To calculate H_0 , observe that G has a single point, and $Im(\partial_1) = Im(D_1) = 0$, since all objects of ∇ go to the identity for a curve $\nabla \rightarrow G$. Therefore $H_0 \xrightarrow{\sim} R$.

To get H_1 , let us look at a curve $F : \nabla \rightarrow \mathbf{1} \overrightarrow{\textcircled{G}}$. It is defined by a natural transformation $\mu(x, y) : F(x) \rightarrow F(y)$ with $\mu(y, z) \circ \mu(x, y) = \mu(x, z)$ for any morphisms $x \leq y, y \leq z$ in ∇ . Thus we have a commutative diagram of group elements for any morphism $s \leq t$, continuous in x, y, s, t , as follows:

$$\begin{array}{ccc} F(x)(s) & \xrightarrow{\mu(x,y)(s)} & F(y)(s) \\ F(x)(s,t) \downarrow & & \downarrow F(y)(s,t) \\ F(x)(t) & \xrightarrow{\mu(x,y)(t)} & F(y)(t) \end{array}$$

where all objects of the square are the identity of G . This implies

$$\mu(x, y)(t) = F(x)(s, t)^{-1} \circ \mu(x, y)(s) \circ F(y)(s, t).$$

For $s = 0$, the natural transformation $\mu(x, y) : F(x) \rightarrow F(y)$ is defined once we have $\mu(x, y)(0) : F(x)(0) \rightarrow F(y)(0)$ such that $\mu(y, z)(0) \circ \mu(x, y)(0) = \mu(x, z)(0)$. But we may set this initial value to Id . Suppose that we are given two curves $F_0, F_1 : \nabla \rightarrow G$. We may define an intermediate curve $F(x) = F_1$ for every $x > 0$. Clearly, then, the natural transformation $\mu(x, 1) = Id$ connects $F(x)$ to F_1 for all $x > 0$. To connect F_0 to $F(x) = F_1$, we take the initial transformation $\mu(0, x) = Id$ and then the following special case of the above formula

$$\mu(0, x)(s) = F_0(0, s)^{-1} \circ Id \circ F(x)(0, s) = F_0(0, s)^{-1} \circ F_1(0, s).$$

It is clear that this rule defines a continuous curve from F_0 to F_1 . This means that any two curves F_0, F_1 are endpoints $F_0 = F(0), F_1 = F(1)$ of a curve $F : \nabla \rightarrow \mathbf{1} \overrightarrow{\textcircled{G}}$, and in turn this means that $Im(D_2)$ is generated by all differences of any two gestures in $\mathbf{1} \overrightarrow{\textcircled{G}}$. Moreover, since all gestures $g \in \uparrow \overrightarrow{\textcircled{G}}$ have $D_1(g) = 0, Ker(D_1) = \uparrow \overrightarrow{\textcircled{G}}$. Therefore $Ker(\partial_1) = C_1(K.c.) = \uparrow \overrightarrow{\textcircled{G}} \oplus R \mathbf{1} \overrightarrow{\textcircled{G}}$. And therefore

$$H_1 \xrightarrow{\sim} R \uparrow \overrightarrow{\textcircled{G}} \oplus R$$

where the second factor represents the quotient of $R \mathbf{1} \overrightarrow{\textcircled{G}}$ modulo the module of gesture differences. The second homology group H_2 is described as in the general case above.

Example 10.4. We now look at the category ${}^G X$ derived from a group action $G \times X \rightarrow X$ as described above. We suppose here that its topology is indiscrete, as in the case of the musical category $\overrightarrow{GL}(\mathbb{Z}_{12}) \mathbb{Z}_{12}$. For the classical cubic homology we have this result:

Theorem 10.1. *The first homology module of an indiscrete topological category ${}^G X$ is the free R -module*

$$H_1({}^G X) = R^{G \setminus X},$$

where $G \setminus X$ is the set of orbits of this action.

The proof resides in a series of easy lemmata, which we leave as an exercise to the reader.

10.2 Homological Interpretation of the Counterpoint Model

Let us first explain why we believe that our mathematical model of counterpoint needs to be reinterpreted in terms of algebraic topology. Let us briefly restate the rules that have been developed in the preceding chapters. We first model intervals as dual numbers $\xi = a + \epsilon k \in \mathbb{Z}_{12}[\epsilon]$, where $k \in K = \{0, 3, 4, 7, 8, 9\}$ is a consonant interval quantity in the ring \mathbb{Z}_{12} of pitch classes. Given a consonance ξ , we select *contrapuntal symmetries* g in $G = \overrightarrow{GL}(\mathbb{Z}_{12}[\epsilon])$, the group of affine automorphisms of $\mathbb{Z}_{12}[\epsilon]$. These are by definition those automorphisms where

1. $\xi \in g(D[\epsilon])$,
2. $g \circ AK_\xi = AK_\xi \circ g$,
3. $g(K[\epsilon]) \cap K[\epsilon]$ is a maximal set with the first two properties.

In this definition, AK_ξ is the unique affine automorphism AK_ξ , which leaves the intervals at cantus firmus pitch class a of ξ invariant 2.2, and recall that $AK_\xi^2 = Id$. The idea is that following classical ideas about contrapuntal motion (see Section 1.3 and [78]), the tension between successive intervals occurs when moving between consonances and dissonances. Since this is impossible, we simulate this desired motion as a transition from *deformed* dissonances to *deformed* consonances, where the deformation is given by the dichotomy $g(K[\epsilon])/g(D[\epsilon])$, i.e., the interval ξ is a deformed dissonance that is succeeded by a deformed consonance. The second requirement in the above rules means that the autocomplementarity AK_ξ is also an autocomplementarity of the deformed dichotomy. The maximality requirement means that we want to reach every possible deformed consonance within a maximal set of choices.

However, this model suffers from the same conceptual deficiency as David Lewin's famous statement in [49] about that "characteristic gesture" when "getting from s to t ." There is no gesture whatsoever in his theory—the continuous gestural movement from s to t is fictional. Lewin's transformations are strictly algebraic; no continuous curves connecting s to t are defined. In our model we recognize a similar deceptive intuition, namely when we speak about those deformed interval sets. The concept of a deformation suggests a continuous movement that successively morphs an interval set into a deformed result. In our setup, similar to Lewin's, we only have symmetries, transformations that don't share any continuous character.

The scope of this chapter is to review our contrapuntal model such that the group-theoretical contrapuntal symmetries are reinterpreted in the framework of topology, where continuity can be addressed. In particular, we shall interpret the set of intervals as being a topological space, and even a topological category. We shall then develop a theory of hypergestures in such a category and investigate the first singular homology group associated with hypergestures. It will turn out that the above

conditions defining contrapuntal symmetries can be restated in terms of topology and its associated homology of hypergestures.

What is the meaning of this result for music theory? It is comparable to the meaning of a result that would enable us to consider Lewin's transformations as being gestures, not only metaphorically speaking, but as real gestures in a topological space. Such a result, and ours is of this type, opens up a topological music theory, a theory that understands music-theoretical rules as consequences of a topological concept framework that invokes homology of hypergestures. We believe that this initiates a considerable paradigmatic shift in music-theoretical thinking. But it also sparks thoughts about a more intimate relationship between thinking and making music, the latter being strongly connected to gestural embodiment.

10.2.1 *Hypergestural Singular Homology*

This section does not deal with general hypergestural singular homology but focuses instead on the classical context of cubic singular homology, meaning that the gestures are all curves $f : \nabla \rightarrow K$ —continuous functors on the simplex category ∇ , with values in a topological category K . Recall that ∇ 's morphism set is $\nabla = \{(x, y) | x, y \in I \text{ and } x \leq y\}$, where $I = \{x | 0 \leq x \leq 1\}$ is the real unit line, the domain is $d(x, y) = (x, x)$, the codomain is $c(x, y) = (y, y)$, the composition of morphisms is obvious, and the topology on ∇ is the relative topology inherited from the usual product topology on $I \times I \subset \mathbb{R} \times \mathbb{R}$. So ∇ is essentially the unit line, enriched with the morphism pairs $(x, y), x \leq y$. Viewed as a categorical gesture [58], such a curve is the representation of the gestural *skeleton* \uparrow as a curve f in the *body* of the category K .

Cubic homology is based upon categories $\uparrow @ \uparrow @ \dots \uparrow @ K$ of n -fold hypergestures in K for the sequence $\uparrow, \dots \uparrow$ of n copies of the one-arrow digraph \uparrow . The difference from classical cubic homology is that here, we look at curves that are also functors, not only continuous.

For the general theory of hypergestures, the yoga of homology is that it formally represents a relationship between neighboring layers of hypergestures over a given topological category. In this chapter we shall study $H_1(R, K)$ for any ring R and for topological categories K related to continuous actions of topological groups on topological spaces.

10.2.2 *A Classical Example of a Topological Category from Counterpoint*

We consider a special topological groupoid ${}^G X$ for a continuous left action $G \times X \rightarrow X$ of a topological group G on a topological space X described in Example 9.5. The first direct example from musical set theory is the indiscrete topological spaces $X = \mathbb{Z}_{12}$, $G = T^{\mathbb{Z}_{12}} \rtimes \mathbb{Z}_{12}^\times$, the pitch classes on which the full affine group acts.

The following example is our test category since it relates intimately to counterpoint theory. The topological space is the space $X = \mathbb{Z}_{12}[\varepsilon]$ of contrapuntal intervals of pitch classes. The topology is defined as follows. Select a consonant interval $\xi = a + \varepsilon k$. Consider the unique affine automorphism AK_ξ of X that leaves the intervals at cantus firmus pitch class a of ξ invariant (Proposition 2.2), and recall that $AK_\xi^2 = Id$. The Kuratowski closure operator of our topology is defined by $\bar{Y} = Y \cup AK_\xi(Y)$. This means that closed subsets $Y \subset X$ are the AK_ξ -invariant subsets. This also implies that $Y \subset X$ is open if and only if it is closed. In particular, if $\eta \in X$ is an interval, its closure is the subset $\bar{\eta} = \{\eta, AK_\xi(\eta)\}$. We call this topology the AK_ξ -topology. It has no closed points, and the set $K[\varepsilon] = \{a + \varepsilon b \mid a \in \mathbb{Z}_{12}, b \in K\}$ ($K = \{0, 3, 4, 7, 8, 9\}$ is the set of Fuxian consonant interval quantities) of all consonant intervals is dense.

The topological group in this example is $G = \overrightarrow{GL}(\mathbb{Z}_{12}[\varepsilon], AK_\xi)$, the group of AK_ξ -topology continuous affine automorphisms of $\mathbb{Z}_{12}[\varepsilon]$. The following lemma shows what it means for an affine automorphism $g : \mathbb{Z}_{12}[\varepsilon] \xrightarrow{\sim} \mathbb{Z}_{12}[\varepsilon]$ to be AK_ξ -continuous.

Lemma 10.1. *An affine automorphism $g : \mathbb{Z}_{12}[\varepsilon] \xrightarrow{\sim} \mathbb{Z}_{12}[\varepsilon]$ is AK_ξ -continuous if and only if it commutes with AK_ξ .*

Proof. If g commutes with AK_ξ , and if $Y \subset X$ is AK_ξ -invariant, then we have $g^{-1}(Y) = g^{-1}(AK_\xi(Y)) = AK_\xi(g^{-1}(Y))$ since g commutes with AK_ξ if and only if g^{-1} does so. Therefore g is continuous. Conversely, if g is continuous, taking the open set $\bar{\eta}$, we get $g^{-1}(\bar{\eta}) = \{g^{-1}(\eta), g^{-1}(AK_\xi(\eta))\}$, which has two elements and is open, meaning that it has the shape $\bar{\tau}$. Now, if $g^{-1}(\eta) = \tau$, then $g^{-1}(AK_\xi(\eta)) = AK_\xi(\tau) = AK_\xi(g^{-1}(\eta))$, and g^{-1} commutes with AK_ξ , so g also commutes with AK_ξ . Else, if $g^{-1}(\eta) = AK_\xi(\tau)$ and $g^{-1}(AK_\xi(\eta)) = \tau$, then $AK_\xi(g^{-1}(\eta)) = \tau = g^{-1}(AK_\xi(\eta))$, and commutation is again true. \square

This elementary topological fact is interesting since the commutation condition is exactly what is required for a so-called contrapuntal symmetry, see Section 10.2. This supports the hope that counterpoint theory could be restated in terms of topology or even in terms of algebraic topology and singular homological algebra.

We now take the group $G = \overrightarrow{GL}(\mathbb{Z}_{12}[\varepsilon], AK_\xi)$ with the compact-open topology. Since $\mathbb{Z}_{12}[\varepsilon]$ is locally compact (it is even finite), the composition of continuous automorphisms is continuous, and we get a topological group.

Lemma 10.2. *With the above notations, the canonical group action $m : G \times X \rightarrow X$ is continuous.*

Proof. It suffices to show that $m^{-1}(\bar{\eta})$ is open for the smallest open sets $\bar{\eta}$. But $m^{-1}(\bar{\eta}) = \{(g, \zeta) \mid g(\zeta) \in \bar{\eta}\}$. In the compact-open topology of G , we have the open sets $[\zeta, \eta] = \{g \mid g(\zeta) \in \bar{\eta}\}$ since ζ is the unique element of the compact singleton $\{\zeta\}$. This means that $m^{-1}(\bar{\eta}) = \{(g, \zeta) \mid g \in [\zeta, \eta]\}$. But we also have $[\zeta, \eta] = [AK_\xi(\zeta), \eta]$; therefore, $m^{-1}(\bar{\eta}) = \bigcup_{\zeta \in X} [\zeta, \eta] \times \bar{\zeta}$, an open set in the product topology. \square

This completes the construction of the topological category ${}^G X = \overrightarrow{GL}(\mathbb{Z}_{12}[\varepsilon], AK_\xi)$ $\mathbb{Z}_{12}[\varepsilon]$ of contrapuntal intervals. In what follows, we shall also consider full subcategories Z of this ${}^G X$. The essential difference is that in those cases, it will not be certain that any group action $g \cdot \eta$ on an object η of Z will automatically yield another object in Z . Nevertheless, for some subcategories, we can prove that they are in fact groupoids ${}^G X$. Here is the lemma that enables this special situation:

Lemma 10.3. *With the above notations, if $\eta \in \mathbb{Z}_{12}[\varepsilon]$ is an interval, then the full subcategory on the open set $\bar{\eta}$ is a groupoid defined by the group $G(\eta)$ generated by the automorphisms that define morphisms of $\bar{\eta}$.*

Proof. The automorphisms of $\bar{\eta}$ are generated by 1) AK_ξ and 2) $Aut(\eta)$. But since AK_ξ and morphisms on $\overrightarrow{GL}(\mathbb{Z}_{12}[\varepsilon], AK_\xi) \mathbb{Z}_{12}[\varepsilon]$ commute by Lemma 10.2, $G(\eta)$ consists of $Aut(\eta)$ and of the products $Aut(\eta) \circ AK_\xi$. □

In the context of continuous groupoids, the following lemma is useful and often used without special mention:

Lemma 10.4. *If $f : \nabla \rightarrow C$ is a curve, then for $0 < \lambda \leq 1$ the concatenation curve $f_\lambda = Id_{f(1)} \cdot_\lambda f$, defined by $f_\lambda(\mu) = f(\mu/\lambda)$ for $\mu \leq \lambda$ and $f_\lambda(\mu) = f(1)$ for $\mu > \lambda$ with the evident transition morphisms admits a morphism of curves $f \rightarrow f_\lambda$. We also have an morphism ${}_\lambda f \rightarrow f$ for the opposite concatenation ${}_\lambda f = f \cdot_\lambda Id_{f(0)}$, defined by the analogous construction. These morphisms between curves are isomorphisms when f is a curve of isomorphisms (all morphisms of the functor are isomorphisms).*

Proof. We define a morphism of curves $q : f \xrightarrow{\sim} g_\lambda$ as follows: For $\mu \leq \lambda$, we map $f(\mu) \rightarrow g_\lambda(\mu) = f(\mu/\lambda)$ by the morphism given from f . For $\mu > \lambda$, we map $f(\mu) \rightarrow g_\lambda(\mu) = f(1)$ by the given morphism from f . The proof of the second statement is analogous, and if f is a curve of isomorphisms, evidently, the curve morphisms are both isomorphisms. □

A general remark on continuous curves $f : \nabla \rightarrow {}^G X$ is important for the above topological category of contrapuntal intervals. If $f(0) = \eta$, then the whole image $f(\nabla)$ must be in the full subcategory induced on $\bar{\eta}$ since the inverse image $f^{-1}(\bar{\eta})$ is a non-empty closed and open subset of ∇ , which can only be all of ∇ .

10.2.2.1 Generators of $H_1({}^G X)$ for the Groupoid ${}^G X$ Defined by a Group Action

We want to describe a set of generators (over any ring R) of the first homology module $H_1({}^G X)$ deduced from the standard (cubic) hypergesture configuration $\uparrow\uparrow \overrightarrow{\textcircled{G}} X$. Here are two standard cycle types in $Z_1({}^G X)$:

Definition 10.3. *Loops and parallel cycles:*

- Loop curves $Loop(x, f) = x \begin{array}{c} \curvearrowright \\ \text{---} f \end{array}$ are defined by loops f starting and ending at objects x .

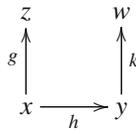
- Pairs of different curves $f \neq g$ from x to $y \neq x$ with opposite signs define these parallel cycles:

$$Para(x, y, f, g) = x \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{-g} \end{array} y .$$

We shall prove the following theorem from a sequence of lemmata and corollaries:

Theorem 10.2. For the standard hypergesture configuration $\uparrow\uparrow \vec{\textcircled{G}}X$ of a continuous groupoid ${}^G X$, the homology module $H_1({}^G X)$ is generated by loop curve cycles in $Z_1({}^G X)$.

Lemma 10.5. For any three curves $g, h, k : \nabla \rightarrow {}^G X$ in this configuration

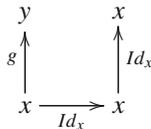


there exists a hypergesture $t \in \uparrow\uparrow \vec{\textcircled{G}}X$ such that $\partial t = g - h - k + l$, with $z \xrightarrow{l} w$ as a fourth curve.

Proof. The construction of t goes in two steps. First, we construct a hypergesture $t_1 \in \uparrow\uparrow \vec{\textcircled{G}}X$ with $t_1(0) = g$ and $t_1(1) = Id_y$, and such that its Escher-corresponding hypergesture starts at h and ends in a gesture m from z to y . Then we define a second hypergesture $t_2 \in \uparrow\uparrow \vec{\textcircled{G}}X$ with $t_2(0) = Id_z$ and $t_2(1) = k$, whose Escher-corresponding hypergesture starts at m and ends at l , a gesture from z to w as required by our theorem. Concatenating t_1 with t_2 yields a hypergesture that starts at g and ends at k , and whose Escher-corresponding hypergesture starts at h and ends at l . This defines the desired hypergesture t . Hypergesture t_1 has the gesture $t_1(\lambda) = {}^\lambda g_{h(\lambda)}$ at curve parameter $\lambda \in I$, where ${}^\tau g(\mu) = g(\tau + (1 - \tau)\mu)$ is the partial gesture of g from τ to 1. And t_2 has gesture $k_{m(\lambda)}^\lambda$ at curve parameter $\lambda \in I$, where k^τ is the partial gesture of k starting at 0 and ending at τ . The gesture morphisms $t_i(\lambda, \mu) : t_i(\lambda) \rightarrow t_i(\mu), i = 1, 2$, for $\lambda \leq \mu$ are evident. \square

Corollary 10.1. Every curve $x \xrightarrow{g} y$ in ${}^G X$ can be reversed modulo loops, i.e., there is a curve $y \xrightarrow{h} x$ such that $g = -h + \partial t + 2Loop(x, Id_x)$.

Proof. Attaching two copies of $Loop(x, Id_x)$ to g at x generates the configuration



which solves our problem in view of Lemma 10.5. \square

This corollary enables us to reverse any curve direction and to focus on cycles with any curve directions we like.

Corollary 10.2. *In the category ${}^G X$, for any two cycles $Loop(x, f), Para(x, y, g, k)$, there is a loop $Loop(y, l)$ such that*

$$Loop(x, f) - Loop(y, l) \equiv Para(x, y, g, k) \pmod{B_1}$$

In particular, any parallel cycle is equivalent to a difference of two loops modulo B_1 .

Lemma 10.6. *In ${}^G X$, modulo parallel arrow cycles $Para(x, y, f, g)$, every cycle in Z_1 is equivalent to a cycle without multiple arrows.*

Proof. If we have two curves c, d between points x, y , Corollary 10.1 enables us to suppose the curves have same direction. Then, if these curves come up with multiplicities γ, δ , we can subtract the γ -fold of the parallel cycle $Para(x, y, c, d)$ from the give cycle and only curve d remains with scalar $\delta + \gamma$. □

And here is the proof of Theorem 10.2:

Proof. Modulo loops, we may suppose that a given cycle $z \in Z_1$ has no multiple arrows or loops. Take a longest closed path p within z . Taking a sequence of three curves in p , we may suppose that they have the directions described in Lemma 10.5. If the factor of the middle curve is λ , we may subtract from p the boundary $\lambda \partial t$ for the t generated in Lemma 10.5. This yields a new closed path p' , which is shorter than p . It is evident that the curve introduced by ∂t does not yield new paths having the length of p . This means that now p' has fewer longest closed paths. We may continue this procedure until in the resulting cycle z' no closed paths of length more than 2 are left, applying Lemma 10.6 if necessary. Finally, following Corollary 10.2, the parallel cycles can be replaced by differences of loops. □

If we take the set of all loop cycles $Loop(x, f)$, they are clearly linearly independent. But are there relations among such loops modulo B_1 ? The next result describes such relations.

Lemma 10.7. *Let $C(K)$ be the set of curve-connected components³ of a topological category K , and let $(x_c)_{c \in C(K)}$ be a family of objects, one from each component $c \in C(K)$. Then for $K = {}^G X$, every cycle $z \in Z_1$ can be represented modulo B_1 by a linear combination of loop cycles $Loop(x_c, f)$.*

Proof. We know from Lemma 10.6 that z is represented by a linear combination of loop cycles $Loop(x_z, f_z)$. Suppose that x_z is in the connected component c containing x_c . Now, choose a curve $x_z \xrightarrow{l} x_c$. Then there is a hypergesture t defined by the configuration

³ Observe that curves are special topological paths in that they must be functors, but not all paths need to be functorial.

$$\begin{array}{ccc}
 & \xrightarrow{Loop(x_c, f)} & \\
 x_c & & x_c \\
 \uparrow l & & \uparrow l \\
 x_z & \xrightarrow{Loop(x_z, f_z)} & x_z
 \end{array}$$

according to Lemma 10.5 such that we have $\partial t = Loop(x_c, f) - Loop(x_z, f_z)$, and therefore $Loop(x_c, f)$ represents $Loop(x_z, f_z)$ on x_c . \square

Lemma 10.8. *With the above notations, every loop cycle $Loop(x_c, f)$ at x_c is equivalent to the constant identity cycle $Id_c = Loop(x_c, Id_{x_c})$, which implies that*

$$H_1(GX) = \bigoplus_{c \in C(GX)} R.Id_c^*$$

Id_c^* being the class of Id_c .

Proof. If we repeat the construction of a hypergesture $t \in \uparrow \vec{\textcircled{G}}X$ from Lemma 10.2 with any loop cycle h at x and the identity loop cycle, we get this diagram

$$\begin{array}{ccc}
 x & & x \\
 \uparrow h & & \uparrow Id_c \\
 x & \xrightarrow{h} & x
 \end{array}$$

and the constructed top horizontal gesture is also h , i.e., $\partial t = h - Id_c$. Since there are no morphisms between different components, the sum is direct. \square

10.2.3 The Meaning of H_1 for Counterpoint

The next step concerns the calculation of homology according to curve-connected components $C(K)$ of topological categories. Since there are no curves connecting different components, the homology module $H_1(K)$ is the direct sum of the homology modules $H_1(c)$ of components $c \in C(K)$. In our contrapuntal example of Section 10.2.2 above, the components are those minimal open two-element sets $\bar{\eta} = \{\eta, AK_\xi(\eta)\}$. Each curve must live in one of these minimal open sets. But these sets bear the indiscrete topology. So the full subcategories $Z = \bar{\eta}$ they induce are exactly of the type we assumed in the preceding results. Calculations of homology can therefore be performed on each of these small subcategories. We may take the representatives $\eta \in K[\varepsilon]$, the set of consonant intervals, and we then obtain the homology

$$H_1(\vec{GL}(\mathbb{Z}_{12}[\varepsilon], AK_\xi) / \mathbb{Z}_{12}[\varepsilon]) = \bigoplus_{\eta \in K[\varepsilon]} H_1(\bar{\eta}).$$

The last step is showing that the classes Id_c^* don't vanish. And we may also suppose that we are working in a fixed curve component, call it W_c .

Lemma 10.9. *With the notation of Lemma 10.8, every class Id_c^* is non-vanishing modulo B_1 .*

Proof. We have to show that an equation $Id_c = \sum_{i=1}^k \lambda_i \partial t_i$ is impossible. We may without loss of generality suppose that all hypergestures t_i are defined at object c ; otherwise define a continuous functor $e : W_c \rightarrow \text{End}(c)$ by conjugation. Take a morphism $f_z : z \rightarrow c$ for each object z of W_c and map a morphism $t : z \rightarrow z'$ to $e(t) = f_w \circ t \circ f_z^{-1}$. As homology commutes with continuous functors, the original equation is conserved, replacing the hypergestures t_i by hypergestures in $\text{End}(c)$. Every boundary ∂t_i is of shape $\partial t_i = a_i + b_i - c_i - d_i$ for curves in C_1 . So we have the equation $Id_c = \sum_{i=1}^k \lambda_i (a_i + b_i - c_i - d_i)$. But the elements a_i, b_i, c_i, d_i are all in the canonical basis of C_1 . Therefore we may define a linear map f sending all a_i, b_i, c_i, d_i different from Id_c to a curve $y \neq Id_c$ and leaving Id_c fixed. If no $y \neq Id_c$ exists, then the boundaries vanish, which contradicts our equation. It is immediately clear that the f -image of such a boundary is either $\pm(Id_c - y)$ or $\pm 2(Id_c - y)$. But no linear combination of such elements can be Id_c . Therefore Id_c does not vanish modulo B_1 . \square

We finally get this description of the first homology module:

Theorem 10.3. *For a topological category ${}^G X$ that is defined as the groupoid of a group action, its first homology module is the direct sum*

$$H_1({}^G X) = \bigoplus_{c \in C(Z)} R.Id_c^*$$

of one-dimensional summands $R.Id_c^$. We therefore have $rk(H_1({}^G X)) = card(C({}^G X))$.*

Corollary 10.3. *We have*

$$H_1(\overrightarrow{GL}(\mathbb{Z}_{12}[\varepsilon], AK_\xi) \mathbb{Z}_{12}[\varepsilon]) \xrightarrow{\sim} R^{K[\varepsilon]},$$

and for a contrapuntal symmetry g ,

$$\dim_R(H_1(\overline{g(K[\varepsilon]) \cap K[\varepsilon]})) = card(g(K[\varepsilon]) \cap K[\varepsilon]).$$

In mathematical counterpoint theory, the intersection $g(K[\varepsilon]) \cap K[\varepsilon]$ is the set of consonances that must be maximized and yields the target intervals starting from the consonance $\xi \notin g(K[\varepsilon])$. According to Corollary 10.3, this maximality is also the maximality of the first homology module of that intersection's closure $\overline{g(K[\varepsilon]) \cap K[\varepsilon]}$. Therefore the *set-theoretical* maximality in fact carries over to a maximality of a *linear dimension of a homology module*. This is what we have been looking for.

10.2.4 Concluding Comments

This result is the consequence of a tricky puzzle between the translation of the commutativity condition for contrapuntal symmetries into a topological statement (they are continuous) and the possibility of reducing homology cycles modulo boundaries such that dimensions are boiled down to a single one per curve component. Observe that despite the small number—two—of elements in such a component $\bar{\eta}$, it contains infinitely many curves. It is therefore not trivial to learn that all of these curves contribute to one and only one homology module dimension.

The crucial point in replacing the maximal set cardinality condition of the previous model by a maximal homology dimension condition lies in the fact that such a dimension is defined even if the underlying set is infinite. This means that this topological approach enables such counterpoint models even when the sets of consonances are infinite. The maximality only relates to the dimension of a module, not its set-theoretical cardinality. This situation may occur when stepping to the colimit of infinite microtonal towers, as in Chapter 8, where we are dealing with infinite interval sets.

Appendix A

Mathematical Basics

A.1 Sets and Relations

We take a somewhat simplified approach to sets in this brief appendix. More details about sets can be consulted in [35] and [34].

Definition A.1. A set S is a collection of objects. An object x within a set S is one of its members, and we write $x \in S$. Two sets S, T are equal, $S = T$, if and only if they have the same elements.

A set S can sometimes be written out explicitly

$$A = \{a_1, a_2, \dots, a_n\}$$

or can be defined *extensively* in terms of a property $P(x)$,

$$A = \{x : P(x) \text{ is true}\}.$$

A set with a finite number k of elements is called a k -set. The 0-set is special.¹ It is called the *empty* set, and it is denoted by \emptyset .

Definition A.2. A set U is a subset of a set S if $x \in U$ implies $x \in S$. We write $U \subseteq S$. Clearly, $S = T$ if and only if $T \subseteq S$ and $S \subseteq T$.

Definition A.3. Given $u \in U$ and $v \in V$, its ordered pair is the set

$$(u, v) := \{\{u\}, \{u, v\}\}.$$

Note that $(u, v) = (x, y)$ if and only if $u = x$ and $v = y$.

Definition A.4. Given two sets U and V , their cartesian product is the set of ordered pairs

$$U \times V := \{(x, y) : x \in U, y \in V\}.$$

¹ Having no elements, any two empty sets are equal and may therefore be identified.

Definition A.5. A relation R between the sets U and V is a subset $R \subseteq U \times V$. If $(a, b) \in R$, we write aRb .

A relation $R \subseteq U \times U$ is said to be

- reflexive if uRu for every $u \in U$.
- symmetric if u_1Ru_2 implies u_2Ru_1 .
- transitive if u_1Ru_2 and u_2Ru_3 implies u_1Ru_3 .

Definition A.6. A relation $R \subseteq U \times U$ is an equivalence relation if it is reflexive, symmetric, and transitive.

For an equivalence relation $R \subseteq U \times U$ we define the subsets

$$[u]_R = \{v \in U : uRv\},$$

which are its *equivalence classes*. These sets define a *partition* of U , which means that their union is U , they are pairwise disjoint (that is, if $[u]_R \neq [v]_R$, then $[u]_R \cap [v]_R = \emptyset$), and none of them is empty if U is not empty (since $u \in [u]_R$). It can be proved that, reciprocally, a partition P of U defines an equivalence relation on U such that its set of equivalence classes (also known as the *quotient set* with respect to R , and denoted by U/R) coincide with P .

Definition A.7. A function between from set U to V is a relation $f \subseteq U \times V$ such that for every $a \in U$, there is a $(a, b) \in f$, and afb_1 and afb_2 implies $b_1 = b_2$.

If $f \subseteq U \times V$ is a function, we also write $f : U \rightarrow V$ or $U \xrightarrow{f} V$. Furthermore, if afb , we write $f(a) = b$. The *preimage* of a subset $S \subseteq V$ under f is the set

$$f^{-1}(S) = \{u \in U : f(u) \in S\}.$$

A function $f : U \rightarrow V$ is

1. *injective* if $f(a_1) = f(a_2)$ implies $a_1 = a_2$,
2. *surjective* if for any $v \in V$ there exists a $u \in U$ such that $f(u) = v$,
3. *bijective* if it is both injective and surjective.

Definition A.8. The powerset of a set S is the set of all its subsets and it is denoted by 2^S .

Definition A.9. Given a set of sets $\mathcal{U} = \{U, V, \dots\}$, we say that it covers a set A or is a cover of A if

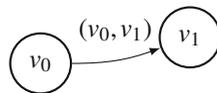
$$A \subseteq \bigcup_{U \in \mathcal{U}} U.$$

A subcover of A is a subset of \mathcal{U} that still covers A .

A.2 Graph Theory

A standard textbook on graph theory is [22]. In the case of counterpoint models, the more application-oriented [33] may turn out to be more useful, especially because it systematically handles digraphs.

Definition A.10. A directed graph D , or digraph for short, is a pair set maps $h, t : A_D \rightarrow V_D$, where V_D is called the vertex set, and A_D the arrow set. The set V_D is non-empty and finite, its elements are called vertices. The set A_D is a possibly empty set, its elements are called arrows. The image vertices $v_1 = h(v)$ and $v_0 = t(v)$ of an arrow v are called its head and tail, respectively. Alternatively, v_0 is also called a predecessor of v_1 and v_1 a successor of v_0 . Often, if there is a single arrow v with tail v_0 and head v_1 , v is also denoted by (v_0, v_1) . We are mostly² dealing with digraphs having at most one arrow from a vertex v_0 to a vertex v_1 . For such digraphs, A_D might then be identified with a subset $A_D \subset V_A \times V_A$. It is tacitly assumed in the following that we deal with this special type of digraphs, although many situations are also valid for general digraphs.



Definition A.11. If E, D are two digraphs, a morphism f from E to D , also denoted by $f : E \rightarrow D$, is a pair of set functions $f = (f_A : A_E \rightarrow A_D, f_V : V_E \rightarrow V_D)$ which commute with both, the head and tail functions. In particular, if both, f_A and f_V are subset inclusions, we say that E is a subdigraph of D .

Definition A.12. Two vertices $v_0, v_1 \in V_D$ are adjacent if they are joined by an arrow, i.e., $(v_0, v_1) \in A_D$. They are non-adjacent if $(v_0, v_1) \in A_D^c$. The arrow set A_D , together with its complement A_D^c form the structure of D .

Definition A.13. A walk in D is an ordered n -tuple of arrows $((v_{i-1}, v_i))_{i=1}^n$, where $(v_{i-1}, v_i) \in A_D$ for any $1 \leq i \leq n$ and $n \in \mathbb{N}$ is the length of the walk.

Definition A.14. A path $((v_{i-1}, v_i))_{i=1}^n$ in D is a walk which does not pass through the same vertex twice, i.e. $v_i \neq v_j$ for $i \neq j$.

Definition A.15. A walk $((v_{i-1}, v_i))_{i=1}^n$ in D forms a cycle if $v_n = v_0$. An n -cycle is a cycle of length n .

Definition A.16. The order of a digraph D is its number of vertices $|V_D|$.

Definition A.17. The size of digraph D is its number of arrows $|A_D|$.

² Except for the final chapter on hypergesture theory.

Definition A.18. The in-degree of a vertex v in D is the number of arrows pointing at v ,

$$d_D^-(v) := \{(v_0, v_1) \in A_D : v_1 = v\}.$$

Similarly, the out-degree of a vertex v in D is the number of arrows leaving v ,

$$d_D^+(v) := \{(v_0, v_1) \in A_D : v_0 = v\}.$$

The total degree of a vertex is the sum of its in- and out-degrees.

Definition A.19. The degree vector of a digraph D is the vector $d \in \mathbb{N}^{|V_D|}$ whose coordinates are given by the degrees of all vertices in non-increasing order.

Definition A.20. The in-neighborhood of a set of vertices $\{v_i\}_{i \in I}$ in D is the set of all predecessors of $\{v_i\}_{i \in I}$,

$$N_D^-(\{v_i\}_{i \in I}) := \{v \in V_D : \exists i \in I \text{ such that } (v, v_i) \in A_D\},$$

while the out-neighborhood is the set of all successors,

$$N_D^+(\{v_i\}_{i \in I}) := \{v \in V_D : \exists i \in I \text{ such that } (v_i, v) \in A_D\}.$$

The in- and out-neighborhoods of a single vertex are simply its predecessors and successors, respectively. Note that this definition allows vertices to be part of their own neighborhoods.³

Definition A.21. A tree T is a digraph in which every vertex has an in-degree equal to one, except one and only one vertex, called the root, whose in-degree equals zero. Vertices of a tree are often called nodes, the (only) predecessor a parent, and successors children. Terminal nodes, i.e., those without children, are called leaves.

Definition A.22. The level of a node v in a tree T is its distance from the root—the length of the path joining them.

Definition A.23. A simple graph G is an ordered pair (V, E) , where V is not empty, and E is a set of 2-element subsets of V , the latter being called edges.

Most of the definitions for digraphs have an almost literal analogue for simple graphs, specifically those of subgraph, adjacency, walks, paths, cycles, order, and size. The definitions of in-degree and out-degree are not distinct for simple graphs, so we speak simply of *degree* of a vertex v , which is the total number of vertices adjacent to v . Similarly, there is only one concept of neighborhood.

Definition A.24. A simple graph is regular if all its vertices have the same degree.

Definition A.25. A simple graph is connected if there is a path between every two of its vertices.

³ Both [33] and [22] exclude $\{v_i\}_{i \in I}$ from its neighborhood. In case of a single vertex, the question arises because of possible loops, but it is not relevant for simple digraphs. See also page 4 in [11].

Definition A.26. A simple graph G is Eulerian if there is a cycle in G that visits all its edges.

Theorem A.1 (Euler-Hierholzer). A simple graph G is Eulerian if and only if it is connected and every vertex has even degree.

Definition A.27. A subgraph H of a graph G is a spanning subgraph or a factor if its set of vertices is the same as G . A k -factor is a regular spanning subgraph such that the degree of all its vertices is k .

A.3 Groups and Rings

Standard introductions to the algebraic structures we discuss here are [86], [77], [37], and [44].

Definition A.28. Let G be a set and $\cdot : G \times G \rightarrow G$ a binary operation. Denote $\cdot(x, y)$ with $x \cdot y$. We say G is a semigroup if it satisfies the following axiom.

Associativity: For $x, y, z \in G$, the identity $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ holds.

Definition A.29. A monoid G is a semigroup that also satisfies the following axiom.

Identity element: There exists $e \in G$ such that $e \cdot g = g \cdot e = g$ for all $g \in G$.

Definition A.30. If S is a set and G is a monoid, then G acts on S (from the left) if there is a function $G \times S \rightarrow S$ (which is the action), denoted by $(g, x) \mapsto gx$, such that for all $g, h \in G$ and $x \in S$,

$$(gh)x = g(hx) \quad \text{and} \quad ex = x,$$

where e is the identity of G .

Given an action of a monoid G on the set S , the set $O(x) = Gx = \{gx : g \in G\}$ is the orbit of x (under the action of G).

Definition A.31. A group G is a monoid that additionally satisfies the following axiom.

Existence of inverses: For any $x \in G$, there exists y such that $x \cdot y = e$.

The inverse of a group element x is always uniquely determined and denoted by x^{-1} , and it is also a left-inverse, i.e., $x^{-1} \cdot x = e$. Note that a group is, in particular, a monoid, thus it also can act on a set.

Example A.1. Given two groups (G_1, \cdot_1) and (G_2, \cdot_2) , we may construct its direct product: it is the set $G_1 \times G_2$ with the operation $(G_1 \times G_2) \times (G_1 \times G_2) \rightarrow G_1 \times G_2$ defined by

$$(g_1, g_2) \cdot_{G_1 \times G_2} (h_1, h_2) = (g_1 \cdot_1 h_1, g_2 \cdot_2 h_2).$$

Definition A.32. If the equation $x \cdot y = y \cdot x$ holds for any x and y in a monoid G , we say G is abelian or commutative.

Definition A.33. A submonoid H of a monoid G is a subset $H \subseteq G$ that is a monoid with the operation \cdot restricted to H , and the neutral elements being the same. In particular, a subgroup H of a group G is a submonoid of G , and then, the neutral elements automatically coincide.

Definition A.34. For an element x in a group G , we define the subgroup generated by x as

$$\langle x \rangle := \{x^i : i \in \mathbb{Z}\},$$

where $x^0 = e$ and $x^{-s} = (x^{-1})^s$ for a positive integer s .

The previous definition can be generalized, taking an arbitrary subset $S \subseteq G$ and calling the smallest subgroup H of G such that $S \subseteq H$ the group *generated by S* . In this situation, we write $H = \langle S \rangle$, and we say the elements of S are the *generators* of G . If S is finite, we say that H is *finitely generated*.

Definition A.35. A ring is a set R and two binary operations $\cdot : R \times R \rightarrow R$ and $+$: $R \times R \rightarrow R$ such that $(R, +)$ is an abelian group, (R, \cdot) is a monoid, and the operation \cdot distributes the operation $+$

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (b + c) \cdot a = b \cdot a + c \cdot a$$

for all $a, b, c \in R$. If (R, \cdot) is an abelian monoid, the ring is said to be commutative.

From this point on, we will refer to an algebraic structure like a semigroup, a group, etc., sometimes simply by its *underlying set* G , and sometimes by tuples like (G, \cdot) to make its operations clear. In addition, given a ring $(R, +, \cdot)$, we will denote with 0 and 1 the respective identities of $(R, +)$ and (R, \cdot) .

Definition A.36. Given a ring $(R, +, \cdot)$, a unit u of the ring is one of its elements such that there exists an $v \in R$ such that $uv = vu = 1$.

The set of units R^\times of a ring R is a multiplicative group, and it is called the *group of units* of R .

Definition A.37. A field F is a ring $(F, +, \cdot)$ such that its group of units F^\times coincides with $F \setminus \{0\}$. In particular, a field cannot be the trivial ring having one single element.

A very important set of examples of algebraic structures are the *cyclic groups*, beginning with the integers \mathbb{Z} (the infinite cyclic group). Another class are the groups \mathbb{Z}_n , whose underlying sets are \mathbb{Z}/R_n , where R_n is defined by

$$aR_n b \quad \text{if and only if} \quad a - b \text{ is divisible by } n.$$

Both types, \mathbb{Z} and \mathbb{Z}_n , are commutative rings. The standard operations $+$ and \cdot are defined for \mathbb{Z}_n by

$$[a]_{R_n} + [b]_{R_n} = [a + b]_{R_n} \quad \text{and} \quad [a]_{R_n} \cdot [b]_{R_n} = [a \cdot b]_{R_n}.$$

These rings are fields if and only if n is a prime number.

A.4 Modules

Definition A.38. A (left) R -module M over a ring R is an abelian group $(M, +)$ endowed with an additional operation $\cdot : R \times M \rightarrow M$ such that

$$1 \cdot m = m, \quad (a + b) \cdot m = a \cdot m + b \cdot m \quad \text{and} \quad a \cdot (m + n) = a \cdot m + b \cdot n$$

for all $a, b \in R$ and $m, n \in M$.

Example A.2. Any abelian group $(G, +)$ is a \mathbb{Z} -module, defining $\cdot : \mathbb{Z} \times G \rightarrow G$ according to

$$\begin{aligned} \cdot : \mathbb{Z} \times G &\longrightarrow G, \\ (n, g) &\longmapsto \begin{cases} \underbrace{g + \cdots + g}_n, & n > 0, \\ 0, & n = 0, \\ \underbrace{(-g) + \cdots + (-g)}_{-n}, & n < 0. \end{cases} \end{aligned}$$

A *homomorphism* is a function $f : A \rightarrow B$ that preserves the algebraic structure of A in B , where algebraic structure is understood as a semigroup, a monoid, a group, a ring, a field, a module, etc. The preservation is understood depending on the specific structure. If (A, \cdot) and $(B, *)$ are semigroups, for example, a monoid homomorphism must satisfy

$$f(a_1 \cdot a_2) = f(a_1) * f(a_2).$$

Another example: If A and B are monoids, a monoid homomorphism is a semigroup homomorphism between them that, additionally, sends the identity of A to the identity of B , i.e., $f(e_A) = e_B$.

Definition A.39. A typology of homomorphisms:

1. An injective homomorphism is a monomorphism.
2. A surjective homomorphism is an epimorphism.
3. A bijective homomorphism is an isomorphism.
4. A homomorphism from a structure to itself is an endomorphism.
5. A bijective endomorphism is an automorphism.

If $(R, M, +_1, \cdot_1)$ and $(R, M, +_2, \cdot_2)$ are two R -modules, a group homomorphism $f : M \rightarrow N$ is called *R -linear* if it is homogeneous, that is,

$$f(r \cdot_1 m) = r \cdot_2 f(m).$$

The set of R -linear homomorphisms from M to N is denoted by $\text{Lin}_R(M, N)$.

Definition A.40. Given two R -modules M and N , an affine homomorphism is a map of the form $T^n \cdot f_0$, where T^n is a translation in N (that is, $T^n(x) = x + n$) and $f_0 \in \text{Lin}_R(M, N)$. An affine endomorphism will also be called an affine transformation or an affine symmetry.

Let p be a prime integer, and $u. = u_1 \leq \dots \leq u_w$ a non-decreasing sequence of positive integers. Set

$$S(p, u.) = \mathbb{Z}_{p^{u_1}} \times \mathbb{Z}_{p^{u_2}} \times \dots \times \mathbb{Z}_{p^{u_w}}.$$

If we are given an increasing sequence $p. = p_1 < p_2 < \dots < p_t$ of primes and for each such p_i a sequence $(u.)_i$, set

$$S(p., (u.)?) = \prod_i S(p_i, (u.)_i).$$

Using this conventions, we can state a very important result.

Theorem A.2 (Classification of Finitely Generated Abelian Groups). For every finitely generated abelian group G , there is a natural number f and a system of primes $p.$ and non-decreasing sequences $(u_i)?$ such that there is an isomorphism

$$G \xrightarrow{\sim} \mathbb{Z}^f \times S(p., (u.)?).$$

The number f and the sequences p_i and $(u_i)?$ are uniquely determined. The length of the sequence $(u.)_i$ is the p_i -rank of G , and the preimage of the factor $S(p., (u.)?)$ under the isomorphism is the torsion group of G . The subgroups corresponding to $S(p_i, (u.)_i)$ are the p_i -Sylow subgroups of G .

A.5 Topology

A good and leisurely paced introduction to general topology is [66]. Another one is [45].

Definition A.41. A topological space is a set X and a collection of its subsets $T \subseteq 2^X$ that satisfy the following axioms:

1. The sets X and \emptyset belong to T .
2. The union of any collection of sets in T belongs to T .
3. The intersection of any finite (non-zero) number of sets in T belong to T .

The elements of T are called open sets, and T is said to be a topology on X .

A useful way to construct topologies for a set X is to take an arbitrary collection of subsets $B \in 2^X$ and then find the smallest topology T such that $B \subseteq T$. With “smallest” we mean that, if S is another topology that contains B , then $T \subseteq S$. In this setup, B is said to generate the topology T .

Definition A.42. A subset K of a topological space X is compact if every open cover (that is, a cover by open subsets of X) of K has a finite subcover.

Definition A.43. A function $f : X \rightarrow Y$ between two topological spaces (X, T) and (Y, S) is said to be continuous if the preimage under f of an open set of Y is an open set of X . In symbols: if $U \in S$, then $f^{-1}(U) \in T$.

Definition A.44. Let X and Y be two topological spaces, and let $C(X, Y)$ denote the set of all continuous functions between X and Y . Given a compact subset K of X and an open set U of Y , let $V(K, U)$ denote the subset of $C(X, Y)$ such that $f(K) \subseteq U$. Then the compact-open topology is the topology generated by the collection of sets $V(K, U)$.

The compact-open topology is a “natural” and common election in order to interpret sets of continuous functions between topological spaces as topological spaces themselves. This is used to define hypergestures, i.e., gestures of gestures, recursively.

A.6 Categories

A very good introduction to the subject is [3]. Both [80] and [14] are nice as well.

A.6.1 Basic Definitions

Definition A.45. A category \mathbf{C} is a collection of objects $\text{Ob}(\mathbf{C})$ such that, for any ordered pair x, y of objects, there is a set—denoted by $\text{Hom}_{\mathbf{C}}(x, y)$, by $\mathbf{C}(x, y)$, or by $x@y$ —of morphisms f , denoted by $f : x \rightarrow y$, that satisfies the following axioms.

1. For each object x , there exists a morphism $\text{Id}_x : x \rightarrow x$.
2. There exists a partial composition law of morphisms⁴ \circ , such that
 - a. If $f : x \rightarrow y$, $g : y \rightarrow z$, then their composition $g \circ f$ is defined, and $g \circ f : x \rightarrow z$.
 - b. It is associative: If $f : x \rightarrow y$, $g : y \rightarrow z$, and $h : z \rightarrow w$ are morphisms, then $f \circ (g \circ h) = (f \circ g) \circ h$.
 - c. The identities of objects act as identities with respect to the composition: $\text{Id}_y \circ f = f$ and $f \circ \text{Id}_x = f$.
 - d. The sets $\text{Hom}_{\mathbf{C}}(x, y)$ are pairwise disjoint. Therefore, for a morphism $f : x \rightarrow y$ in a category, the object x is well-defined and is called its domain and similarly, y its codomain.

⁴ This means that it may not be defined for some pairs of morphisms.

If for every pair of morphisms $g, h : z \rightarrow x$ the equation

$$f \circ g = f \circ h$$

implies $g = h$ (that is, we can cancel f on the left), then f is a *monomorphism*. This fact is shown the notation writing $f : x \rightarrow y$. If for every pair of morphisms $g, h : y \rightarrow z$ the equation

$$g \circ f = h \circ f$$

implies $g = h$ (that is, we can cancel f on the right), then f is an *epimorphism*. This fact is shown in the notation writing $f : x \rightarrow y$.

It is well known that if we attempt to treat the collection of all sets like a set we are led to logical contradictions (a famous one is the *Russell paradox*), but in category theory we must somehow deal with these kind of agglomerations. Thus, we distinguish between sets proper and *classes*, where sets are always classes but not the converse. A category such that its class of objects is a set is called *small*. For the sake of simplicity, we also talk about functions between classes to refer to rules to associate objects, even when they are not sets.

Example A.3. The category **Set** has the class of sets as objects (thus $\text{Ob}(\mathbf{Set})$ is not a set) and the functions between them as morphisms.

Example A.4. For two categories **A**, **B**, their direct product $\mathbf{A} \times \mathbf{B}$ is the category, whose objects are the ordered pairs (x, y) of objects x in **A** and y in **B**, whereas the morphisms $(x, y) \rightarrow (u, v)$ are pairs $(f : x \rightarrow u, g : y \rightarrow v)$ with the evident factor-wise composition law and identities.

Example A.5. Given a category **C** and one of its objects $C \in \text{Ob}(\mathbf{C})$, the *constant category* $\ulcorner C \urcorner$ has C as its only object and Id_C as its only morphism. Note that this is a small category.

Example A.6. For each category **C** we have the *opposite category* \mathbf{C}^{op} , which has the same morphisms and objects but its composition is given by $f \circ^{\text{op}} g = g \circ f$. In \mathbf{C}^{op} , domains and codomains are exchanged, i.e., if $f : x \rightarrow y$ is a morphism in **C**, then, in \mathbf{C}^{op} , it is the morphism $f : y \rightarrow x$.

Example A.7. Given a category **C** and an object I in it, the category \mathbf{C}/I of “objects over I ” has as objects the morphisms $f : X \rightarrow I$ and its morphisms $h : f \rightarrow g$ are the morphisms h in **C** such that $g \circ h = f$. That is, the triangle

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ & \searrow f & \swarrow g \\ & & I \end{array}$$

commutes.

Definition A.46. Let **C** and **D** be categories. A (covariant) functor T from **C** to **D** (written $F : \mathbf{C} \rightarrow \mathbf{D}$) is a function that assigns to each morphism f in **C** a morphism $F(f)$ in **D** that satisfies the following:

1. If f is an identity, so is $F(f)$. Therefore, F defines also a function from the object class of \mathbf{C} to the object class of \mathbf{D} .
2. If $f \circ g$ is defined, so is $F(f) \circ F(g)$ and

$$F(f \circ g) = F(f) \circ F(g).$$

A contravariant functor between the categories \mathbf{C} and \mathbf{D} is a covariant functor between \mathbf{C}^{op} and \mathbf{D} .

Definition A.47. If $F, G : \mathbf{C} \rightarrow \mathbf{D}$ are two functors, a natural transformation $t : F \rightarrow G$ is a system of morphisms $t_c : F(c) \rightarrow G(c)$ in \mathbf{D} , one for each object c of \mathbf{C} , such that for each object c in \mathbf{C} and each morphism $f : x \rightarrow y$ in \mathbf{C} , the following holds

$$G(f) \circ t_x = t_y \circ F(f),$$

in other words, the diagram

$$\begin{array}{ccc} F(x) & \xrightarrow{t_x} & G(d) \\ F(f) \downarrow & & \downarrow G(f) \\ F(y) & \xrightarrow{t_y} & G(y) \end{array}$$

commutes.

Definition A.48. The class of all the contravariant functors from \mathbf{C} to \mathbf{Set} , with the natural transformations between such functors as morphisms, is called the category of presheaves and it is denoted by $\mathbf{C}^{\text{@}}$.

Definition A.49. Let \mathbf{C} and \mathbf{I} be categories, where \mathbf{I} is small. A diagram Γ (in \mathbf{C}) is a covariant functor $\Gamma : \mathbf{I} \rightarrow \mathbf{C}$. If the set of morphisms in the domain \mathbf{I} of a diagram is finite, the diagram is said to be finite.

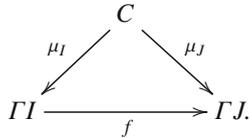
Example A.8. A typical category \mathbf{I} for diagrams is the *path category* $\mathbf{I} = \text{Path}(G)$ of a digraph G . Its objects are the vertices of G , while the morphisms are the paths, i.e., what was called “walks” in this book (the standard name is however “path”). The composition of morphisms is the concatenation of paths. The lazy paths (no arrows, just the vertex) are the identities. A diagram from such a path category in \mathbf{C} is defined by assigning to every vertex an object of \mathbf{C} and to every arrow of G a morphism in \mathbf{C} that connects the objects assigned to its vertices.

Definition A.50. A cone (a cocone) (C, μ) over the diagram $\Gamma : \mathbf{I} \rightarrow \mathbf{C}$ consists of an object $C \in \mathbf{C}$ and a natural transformation $\mu : \ulcorner \mathbf{C} \urcorner \rightarrow \Gamma$ (a natural transformation $\mu : \Gamma \rightarrow \ulcorner \mathbf{C} \urcorner$).

For a cone C , the following diagram must commute

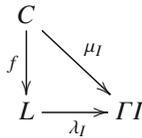
$$\begin{array}{ccc}
 \Gamma C \cap I = C & \xrightarrow{\mu_I} & \Gamma I \\
 \downarrow Id_C & & \downarrow \Gamma f \\
 \Gamma C \cap J = C & \xrightarrow{\mu_J} & \Gamma J
 \end{array}$$

for each $f : I \rightarrow J$, but this is equivalent to the commutativity of the following diagram, which explains the name *cone*,



The *cocone* situation is similar, the arrows at C now having C as their codomain.

Definition A.51. We say a cone (L, λ) over $\Gamma : \mathbf{I} \rightarrow \mathbf{C}$ is a limit cone (or simply limit) if it has the following universal property: For any cone (C, μ) over Γ , there exists a unique morphism $C \xrightarrow{f} L$ such that for any $I \in \mathbf{I}$ the diagram



commutes. We denote the limit cone (L, λ) by $\lim \Gamma$. The analogous situation for a cocone defines a colimit, denoted by $\text{colim} \Gamma$.

Limits generalize cartesian products (with their projection arrows), whereas colimits generalize disjoint unions (with their injection arrows).

Definition A.52. If in a category every diagram has a limit cone (a limit cocone), it is called complete (cocomplete). Analogously, if every finite diagram has a limit cone (a limit cocone), the category is said to be finitely complete (finitely cocomplete).

A finitely complete category has the special limit of the empty diagram, this is the final object 1 , it is characterized by the property that there is a unique morphism $! : X \rightarrow 1$ from each object X of the category. For the category of sets, this is the singleton set $1 = \{\emptyset\}$. For any two objects A, B in such a category, the diagram defined by the two morphisms $A \rightarrow 1, B \rightarrow 1$ is called their *direct or cartesian product* and denoted by $A \times B$.

A.6.2 Subfunctors and Sieves

Let F in $\mathbf{C}^{\textcircled{a}}$. A functor G in $\mathbf{C}^{\textcircled{a}}$ is a *subfunctor* of F if $G(X) \subseteq F(X)$ for every X in $\text{Ob}(\mathbf{C})$ and

$$\begin{array}{ccc} G(X) & \xrightarrow{i_X} & F(X) \\ Gf \downarrow & & \downarrow Ff \\ G(Y) & \xrightarrow{i_Y} & F(Y) \end{array}$$

commutes for every morphism f in \mathbf{C} .

For example, for a subfunctor R of a presheaf $\text{Hom}_{\mathbf{C}}(?, X) = @X$ we have $R(Y) \subseteq \text{Hom}_{\mathbf{C}}(Y, X) = Y@X$. When $u : Y \rightarrow X$ belongs to $R(Y)$, we say that $u \in R$. The class of all the $u \in R$ has the following property:

$$\text{If } u : Y \rightarrow X \in R \text{ and } v : Z \rightarrow Y, \text{ then } u \circ v \in R, \tag{A.1}$$

since $(R(u))(v) = u \circ v \in R(Z)$.

A class R of morphisms with codomain X that possesses the property (A.1) is called a *sieve* of X . A sieve determines uniquely a subfunctor of $@X$ defining

$$R(Y) = \{u \in R : \text{dom}(u) = Y\}$$

and for $v : Z \rightarrow Y$,

$$\begin{aligned} R(v) : R(Y) &\longrightarrow R(Z), \\ u &\longmapsto u \circ v. \end{aligned}$$

A.6.3 Subobjects and Object Classifiers

Let \mathbf{C} be a category and take $I \in \text{Ob}(\mathbf{C})$. Among the category \mathbf{C}/I from Example A.7, let us consider only those objects that are monomorphisms, $A \rightarrowtail I$. Suppose that two objects $f : A \rightarrowtail I, g : B \rightarrowtail I$ over I are related by an isomorphism $h : f \xrightarrow{\sim} g$. This relation over these monomorphism over I is an equivalence relation, and each equivalence class is called a *subobject* of I .

Definition A.53. In a finitely complete category \mathbf{C} , a subobject classifier is a monomorphism $v : 1 \rightarrow \Omega$ such that, for each monomorphism $S \rightarrowtail X$ in \mathbf{C} , there exists a unique χ such that we may construct a pullback

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ m \downarrow & & \downarrow v \\ X & \xrightarrow{\chi_S} & \Omega, \end{array} \tag{A.2}$$

which means by definition of a pullback that S is the limit of the diagram defined by the two morphisms between X , Ω , and 1 .

From the diagram (A.2) it can be deduced that each morphism $X \rightarrow \Omega$ determines uniquely a subobject of X . In fact, since v is a monomorphism, its pullback m must also be a monomorphism. Denoting by $\text{Sub}_{\mathbf{C}}(X)$ the set of all the subobjects of X , we have

$$\text{Sub}_{\mathbf{C}}(X) \xrightarrow{\sim} \text{Hom}_{\mathbf{C}}(X, \Omega).$$

A.6.4 Adjoint Functors

Definition A.54. Given two functors $F : \mathbf{A} \rightarrow \mathbf{B}$ and $G : \mathbf{B} \rightarrow \mathbf{A}$, we say that G is the left adjoint of F if there exists a natural equivalence, i.e., bijections $\theta_{A,B}$ of sets

$$\theta_{A,B} : \text{Hom}_{\mathbf{A}}(GB, A) \rightarrow \text{Hom}_{\mathbf{B}}(B, FA)$$

for all objects A in \mathbf{A} and B in \mathbf{B} that define a natural transformation $\theta : \mathbf{B}^{\text{op}} \times \mathbf{A} \rightarrow \mathbf{Set}$. We also say that F is the right adjoint of G .

In a finitely complete category \mathbf{C} , an object A defines the cartesian product functor $A \times ? : X \mapsto A \times X$. If this functor has a right adjoint $?^A$, A is said to be *exponentiable*. In symbols, we have the natural bijections

$$\text{Hom}_{\mathbf{C}}(A \times X, Y) \xrightarrow{\sim} \text{Hom}_{\mathbf{C}}(X, Y^A)$$

for all X and Y in \mathbf{C} .

A.6.5 Topoi

A *topos* is a special type of category that has most of the features of the category of sets, in the sense that many useful mathematical constructions are available within the category (finite limits, finite colimits, and the like).

Definition A.55. A category \mathbf{C} is an (elementary) topos if it has finite limits and colimits, every object is exponentiable, and it has a subobject classifier.

Example A.9. The category \mathbf{Set} is a topos. The subobject classifier is the injection $1 \mapsto \Omega = \{\emptyset, 1\}$ that maps $\emptyset \in 1$ to itself. The function χ is the well known characteristic function of a subset.

Theorem A.3. For any category \mathbf{C} , the category \mathbf{C}^{op} is a topos.

Theorem A.4. The category \mathbf{C}/I is a topos whenever \mathbf{C} is a topos.

Appendix B

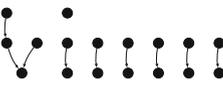
A Guide to Counterpoint Worlds

Different counterpoint worlds can share the same structure, the homogeneous digraphs being isomorphic. The following tables list the different classes of counterpoint worlds along with some graph parameters. The counterpoint worlds \mathcal{A} are grouped by homogeneous structure H . The next columns show the number of tones $2k$, the consonances X , the number of connected components $|V_W|$, and the number of homogeneous components $|V_H|$ and arrows $|A_H|$.

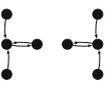
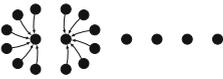
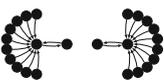
B.1 Discrete Digraphs

| H | Δ | $2k$ | X | $ V_W $ | $ V_H $ | $ A_H $ |
|-----------|----------|------|--------------------|---------|---------|---------|
| • | 2 | 2 | 0 | 1 | 1 | 1 |
| | 279 | 14 | 0 1 2 3 4 5 7 | 3 | 3 | 3 |
| | 296 | | 0 1 2 3 5 7 9 | | | |
| | 311 | | 0 1 2 3 5 10 11 | | | |
| | 327 | | 0 1 2 4 5 7 10 | | | |
| | 370 | | 0 1 2 5 6 9 10 | | | |
| | 408 | | 0 1 3 6 7 10 11 | | | |
| • • • | 907 | 16 | 0 1 2 3 4 5 7 9 | | | |
| | 1009 | | 0 1 2 3 5 7 9 14 | | | |
| | 1237 | | 0 1 2 5 6 8 11 12 | | | |
| | 1246 | | 0 1 2 5 6 10 11 12 | | | |
| | 1300 | | 0 1 3 4 7 9 10 13 | | | |
| | 1327 | | 0 1 3 5 7 9 12 14 | | | |
| | 1339 | | 0 1 3 6 7 9 12 13 | | | |
| | 39 | 10 | 0 1 2 5 6 | 4 | 4 | 4 |
| • • • • | 45 | | 0 1 3 5 8 | | | |
| | 298 | 14 | 0 1 2 3 4 5 11 | | | |
| • • • • • | 7 | 6 | 0 1 3 | 5 | 5 | 5 |
| | 932 | 16 | 0 1 2 3 4 6 7 13 | 8 | 8 | 8 |
| • • • • | 975 | | 0 1 2 3 5 6 7 12 | | | |
| • • • • | 1110 | | 0 1 2 4 5 7 11 14 | | | |
| | 1159 | | 0 1 2 4 6 7 11 13 | | | |
| • • • • • | 363 | 14 | 0 1 2 4 8 9 11 | 9 | 9 | 9 |
| • • • • | | | | | | |
| • • • • | | | | | | |
| • • • • | | | | | | |
| • • • • | | | | | | |
| • • • • | 320 | 14 | 0 1 2 3 7 8 9 | 11 | 11 | 11 |
| • • • • | 390 | | 0 1 3 5 7 9 11 | | | |
| • • • | 404 | | 0 1 3 5 8 9 11 | | | |

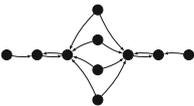
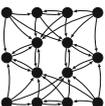
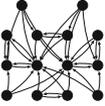
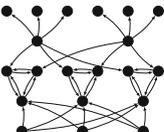
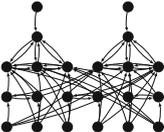
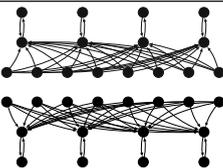
B.2 Forests

| H | Δ | $2k$ | X | $ V_W $ | $ V_H $ | $ A_H $ |
|-----------------------------------------------------------------------------------|-------------------|------|----------------------------------------------------|---------|---------|---------|
|  | 33 46 | 10 | 0 1 2 3 5 0 1 3 6 7 | 3 | 4 | 5 |
|  | 64 | 12 | 0 1 2 3 4 6 | 3 | 8 | 13 |
|  | 68 | 12 | 0 1 2 3 5 8 | 4 | 9 | 14 |
|  | 36 44 | 10 | 0 1 2 4 6 0 1 3 5 7 | 7 | 12 | 17 |
|  | 346 348 401 | 14 | 0 1 2 4 6 8 10 0 1 2 4 6 8 12 0 1 3 5 7 9 12 | 1 | 17 | 26 |

B.3 Stars

| H | Δ | $2k$ | X | $ V_W $ | $ V_H $ | $ A_H $ |
|-------------------------------------------------------------------------------------|------------------------------|------|----------------------------------------------------------------------------------|---------|---------|---------|
|  | 13 16 | 8 | 0 1 2 4 0 1 3 5 | 2 | 8 | 16 |
|  | 82 | 12 | 0 1 2 5 6 9 | 6 | 18 | 30 |
|  | 1169 1171 1176 1324 | 16 | 0 1 2 4 6 8 10 12 0 1 2 4 6 8 10 14 0 1 2 4 6 8 12 14 0 1 3 5 7 9 11 13 | 2 | 24 | 48 |

B.4 Grids

| H | Δ | $2k$ | X | $ V_W $ | $ V_H $ | $ A_H $ |
|-------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|------|--------------------|---------------|---------|---------|
|  | 71 | 12 | 0 1 2 3 6 7 | 1 | 10 | 24 |
| | 965 | 16 | 0 1 2 3 4 8 9 10 | | | |
| | 1020 | | 0 1 2 3 5 8 9 11 | | | |
| | 1040 | | 0 1 2 3 5 10 11 13 | | | |
| | 1062 | | 0 1 2 3 6 8 10 11 | | | |
| | 1078 | | 0 1 2 3 7 9 10 11 | | | |
| | 1204 | | 0 1 2 4 7 9 10 12 | | | |
| | 1281 | | 0 1 3 4 6 8 11 14 | | | |
| | 1309 | | 0 1 3 5 6 8 11 13 | | | |
|  | 929 | 16 | 0 1 2 3 4 6 7 10 | 1 | 12 | 36 |
| | 951 | | 0 1 2 3 4 6 12 13 | | | |
| | 974 | | 0 1 2 3 5 6 7 11 | | | |
| | 1017 | | 0 1 2 3 5 7 12 13 | | | |
| | 1107 | | 0 1 2 4 5 7 10 14 | | | |
| | 1187 | | 0 1 2 4 6 10 11 13 | | | |
| | 1255 | | 0 1 2 5 7 9 11 12 | | | |
| | 1277 | | 0 1 3 4 6 8 10 13 | | | |
| |  | 283 | 14 | 0 1 2 3 4 6 8 | 1 | 12 |
| 352 | | | 0 1 2 4 6 10 11 | | | |
| 405 | | | 0 1 3 5 7 9 12 | | | |
|  | 288 | 14 | 0 1 2 3 4 7 8 | 1 | 14 | 54 |
| | 301 | | 0 1 2 3 5 7 9 | | | |
| | 355 | | 0 1 2 4 7 8 10 | | | |
| | 375 | | 0 1 2 5 7 9 10 | | | |
| | 383 | | 0 1 3 4 7 8 11 | | | |
| | 403 | | 0 1 3 5 7 10 12 | | | |
|  | 304 | 14 | 0 1 2 3 5 7 12 | 1 | 14 | 54 |
| | 316 | | 0 1 2 3 6 8 9 | | | |
| | 333 | | 0 1 2 4 5 8 12 | | | |
| | 362 | | 0 1 2 4 8 9 10 | | | |
| | 402 | | 0 1 3 5 7 10 11 | | | |
| | 409 | 16 | 0 1 2 3 4 5 6 8 | | | |
|  | 75 | 12 | 0 1 2 4 5 8 | 1 | 20 | 59 |
| | | | | | | |
|  | 78 | 12 | 0 1 2 4 6 10 | 1 | 22 | 96 |
| | | | | | | |
|  | 1084 | 16 | 0 1 2 4 5 6 8 12 | 2 | 32 | 112 |
| | 1127 | | 0 1 2 4 5 8 12 14 | | | |
| | 1175 | | 0 1 2 4 6 8 12 13 | | | |
| | 1326 | | 0 1 3 5 7 9 12 13 | | | |

B.5 Unknown Digraphs

| H | Δ | $2k$ | X | $ V_W $ | $ V_H $ | $ A_H $ |
|-------|----------|------|--------------------|---------|---------|---------|
| $?^a$ | 918 | 16 | 0 1 2 3 4 5 8 9 | ? | ? | ? |
| | 935 | | 0 1 2 3 4 6 8 10 | | | |
| | 939 | | 0 1 2 3 4 6 8 14 | | | |
| | 957 | | 0 1 2 3 4 7 9 10 | | | |
| | 980 | | 0 1 2 3 5 6 8 11 | | | |
| | 988 | | 0 1 2 3 5 6 9 14 | | | |
| | 1006 | | 0 1 2 3 5 7 9 11 | | | |
| | 1039 | | 0 1 2 3 5 10 11 12 | | | |
| | 1052 | | 0 1 2 3 6 7 10 11 | | | |
| | 1101 | | 0 1 2 4 5 7 9 12 | | | |
| | 1116 | | 0 1 2 4 5 8 9 12 | | | |
| | 1126 | | 0 1 2 4 5 8 12 13 | | | |
| | 1155 | | 0 1 2 4 6 7 10 12 | | | |
| | 1186 | | 0 1 2 4 6 10 11 12 | | | |
| | 1198 | | 0 1 2 4 7 8 10 14 | | | |
| | 1219 | | 0 1 2 4 8 9 11 13 | | | |
| | 1220 | | 0 1 2 4 8 9 12 13 | | | |
| | 1247 | | 0 1 2 5 6 10 11 13 | | | |
| | 1259 | | 0 1 2 5 8 9 11 12 | | | |
| | 1290 | | 0 1 3 4 6 9 12 13 | | | |
| | 1325 | | 0 1 3 5 7 9 11 14 | | | |
| | 1328 | | 0 1 3 5 7 10 11 13 | | | |
| | 1334 | | 0 1 3 5 8 9 12 13 | | | |
| | 1337 | | 0 1 3 5 8 10 11 14 | | | |

^a The computer's memory capacity was not sufficient for building the digraphs and their quotient structures. These digraphs may not all share the same structure.

Appendix C

Strict Digraphs

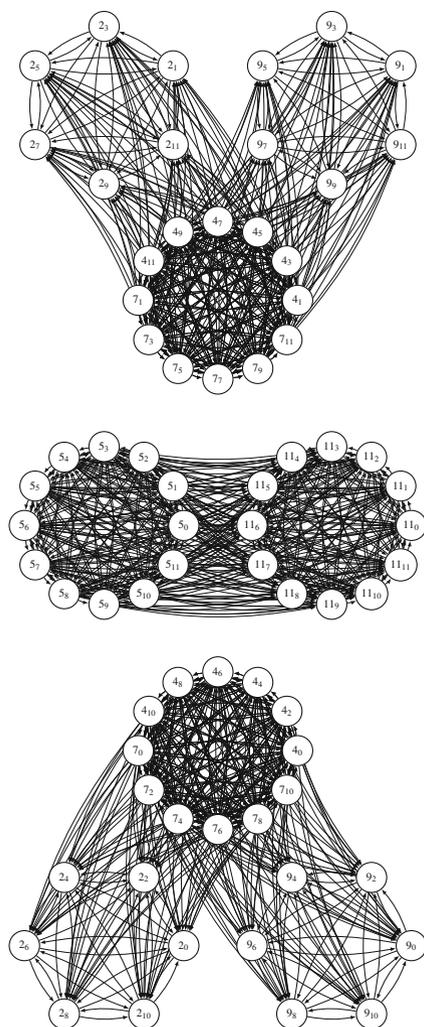


Fig. C.1 Strict digraph of twelve-tone world number 64 (Ionian) in [57].

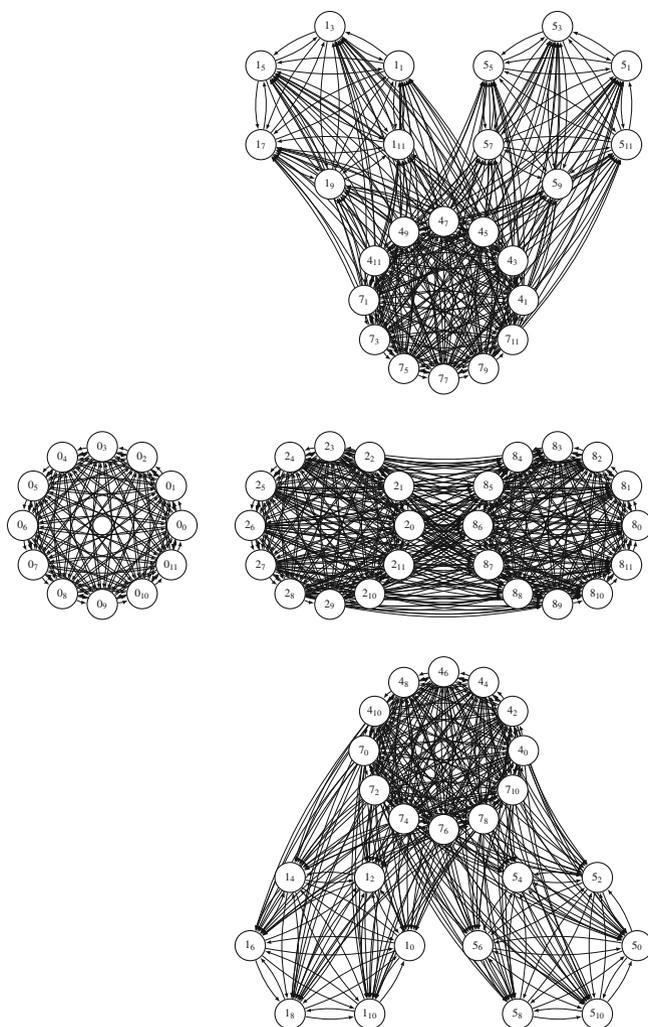


Fig. C.2 Strict digraph of twelve-tone world number 68 in [57]. All numbers are residue classes modulo 12. For the sake of readability, loops are not displayed.

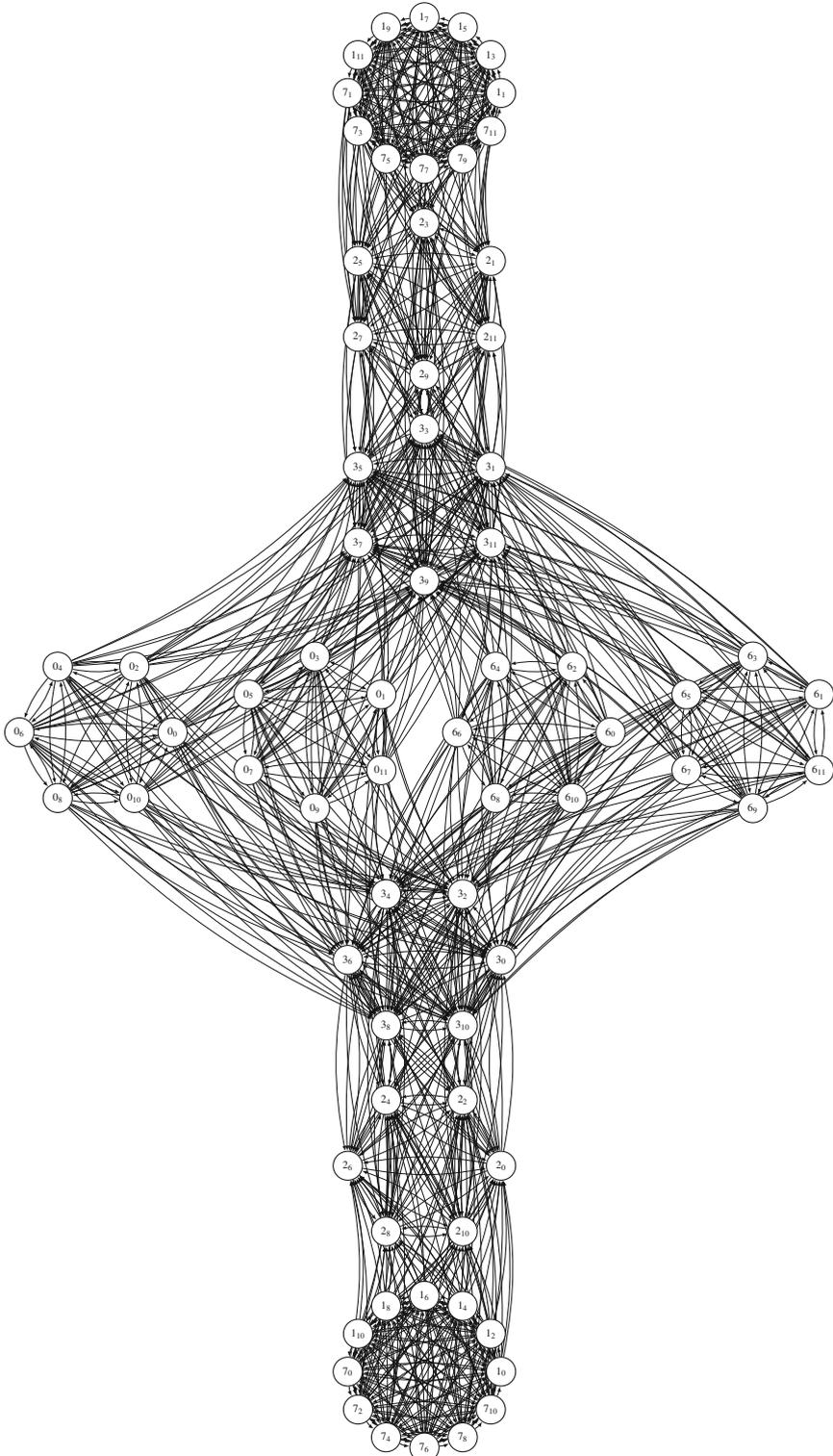


Fig. C.3 Strict digraph of twelve-tone world number 71 in [57]. All numbers are residue classes modulo 12. For the sake of readability, loops are not displayed.

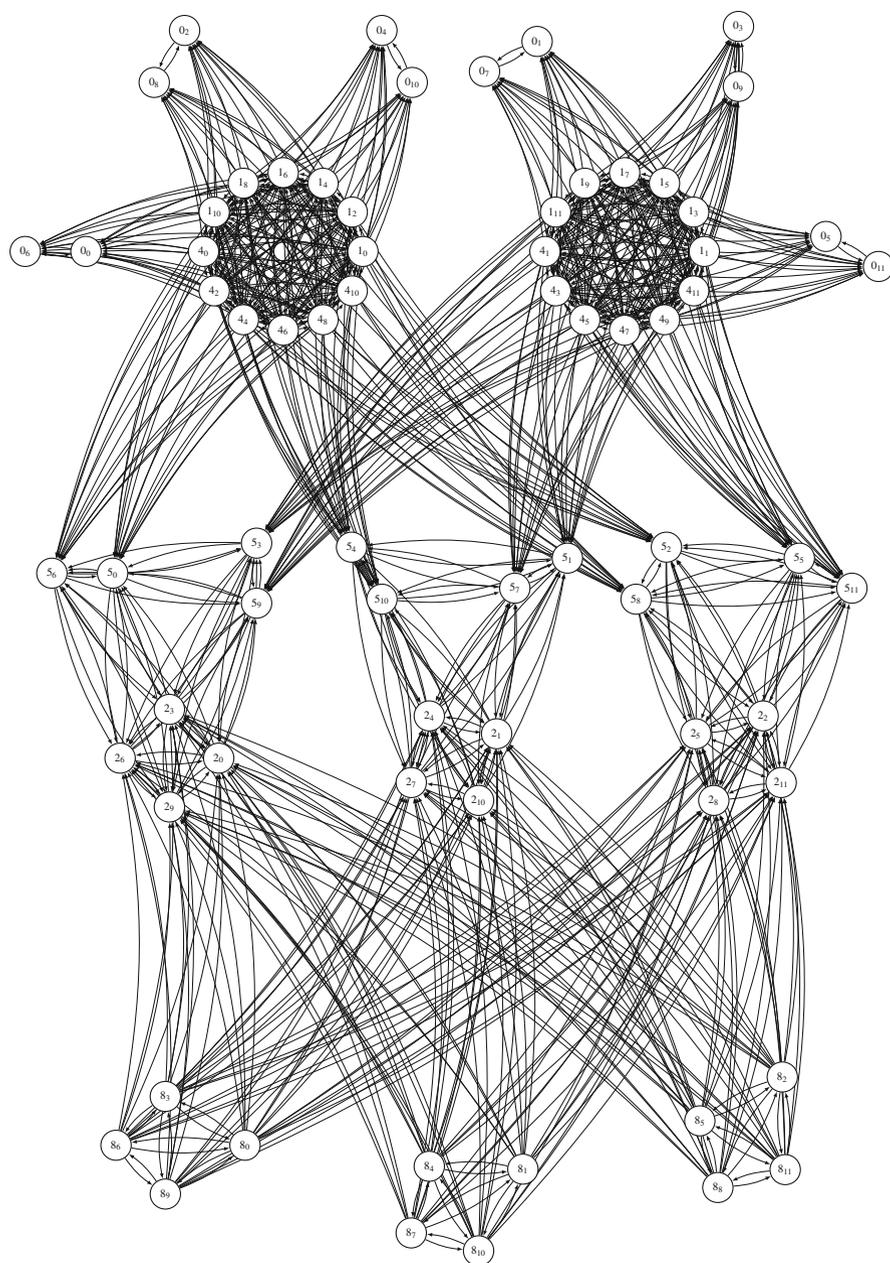


Fig. C.4 Strict digraph of twelve-tone world number 75 in [57]. All numbers are residue classes modulo 12. For the sake of readability, loops are not displayed.

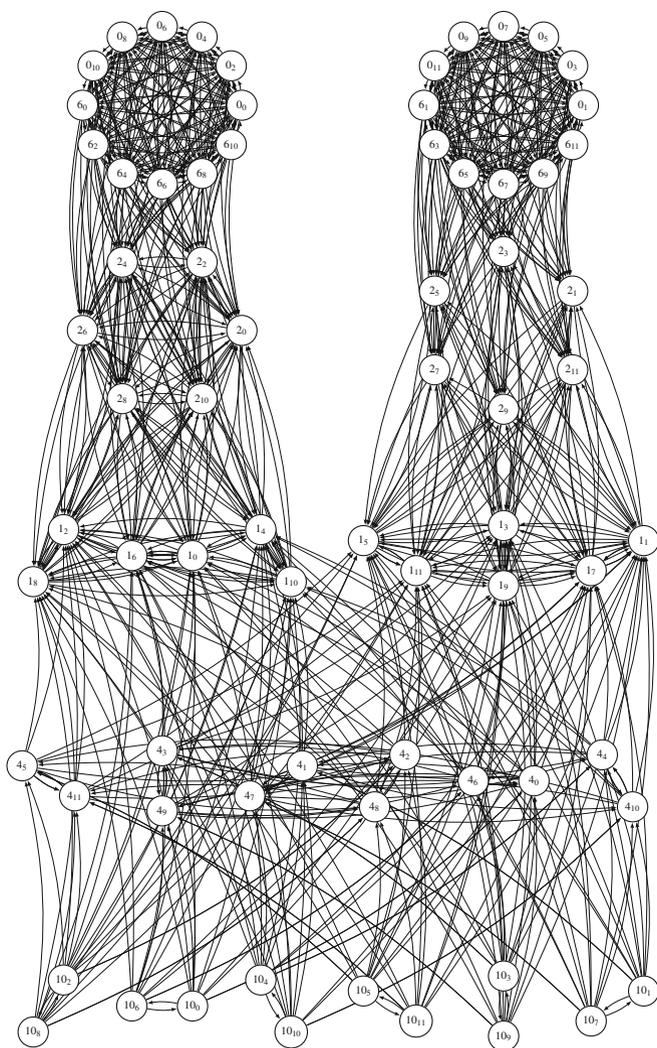


Fig. C.5 Strict digraph of twelve-tone world number 78 in [57]. All numbers are residue classes modulo 12. For the sake of readability, loops are not displayed.

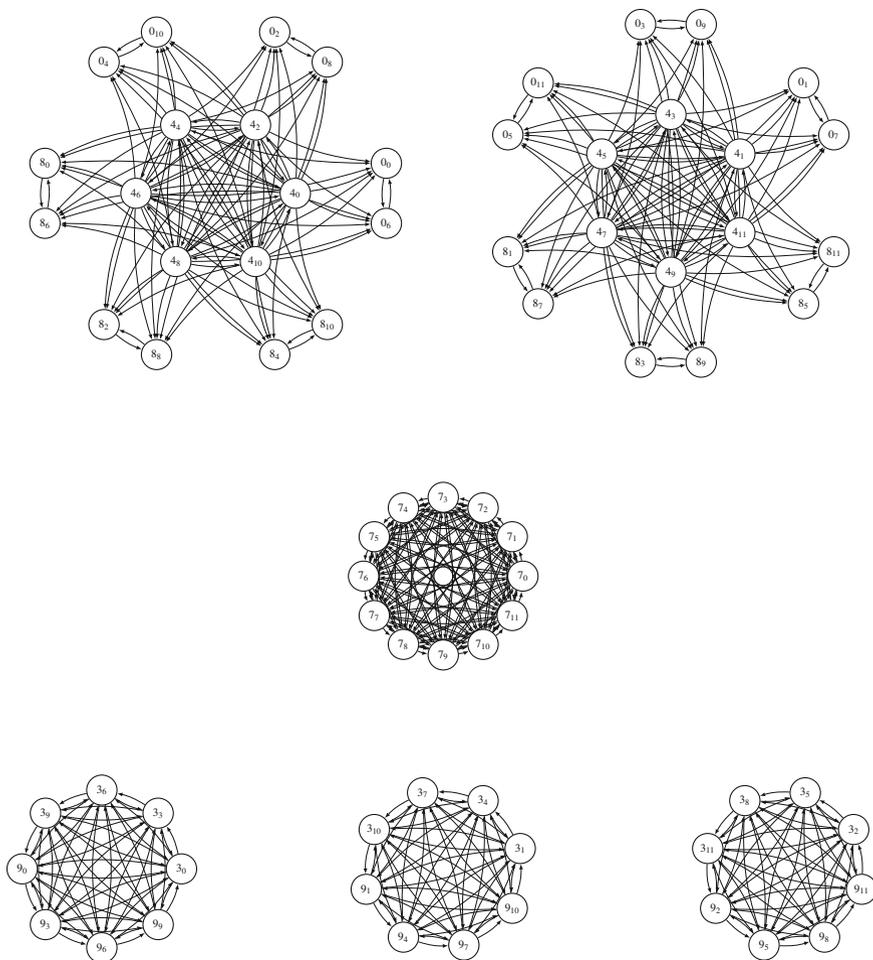


Fig. C.6 Strict digraph of twelve-tone world number 82 (Fuxian) in [57]. All numbers are residue classes modulo 12. For the sake of readability, loops are not displayed.

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